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MONTE CARLO METHODS IN NONLINEAR FILTERING THEORY

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MONTE CARLO METHODS IN NONLINEAR FILTERING THEORY

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To my grandfather Alfredo,
a father who has inspired me and
keeps motivating me to go further.
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“Frodo Baggins: – I wish the Ring had never come to me.
           I wish none of this had happened.
Gandalf the Grey: – So do all who live to see such times.
           But that is not for them to decide.
All we have to decide is what to do with the time that is given to us.”
           — J.R.R. TOLKIEN
(Lord of the rings: the fellowship of the ring)

“Nothing in life is to be feared.
It is only to be understood.”
           — MARIE CURIE
Abstract

This thesis is focused on two basic aspects of the Control Problem: the stochastic modelling of physical systems, and Monte Carlo-based numerical approximation of the nonlinear filtering problem solution. In the first topic this thesis concerns about clarifying some issues in the mathematical modelling of continuous-time systems with Brownian motion. The hypothesis that physical systems should be modelled in continuous-time approach is defended, once the main results in Physics provide solutions for dynamic systems via continuous-time differential equations. It was shown, recalling a main result from the 1960’s that a physical system is represented by Fisk-Stratonovich stochastic differential equation, though Itô approach is better to manipulate the mathematical operations. The required conditions for implementing these equations in computers were also studied by using Euler-Maruyama and Milstein schemes of discretization. In the second topic a unified treatment of the available Monte Carlo methods solving the nonlinear filtering problem for continuous and discrete-time modelling is presented with sufficient emphasis on basic applications enabling the engineer to use results provided by the theory. This topic is branched in the study of the theory of nonlinear filtering problem in continuous and discrete-time approaches, and in the investigation of the aspects of Monte Carlo-based numerical solutions approximating unnormalized conditional expectations, as those given by the classical Kallianpur-Striebel formula and its derived robust representation. Investigations showed that the estimates obtained via numerical approximations of the robust representation, or pathwise filter, might accumulate errors when the observation makes this filter alternative equation unstable, a limitation of the method. Another result of this thesis refers to the implementation of Monte Carlo filters using Bayesian representation for discretized models. Although Monte Carlo methods are attractive due to their facility of parallelization, their main drawback is the degeneracy phenomenon of the particles. The traditional resampling scheme solves the problem, but it difficulties the parallelization of the algorithm. The restoration method was then proposed to move the particles towards higher regions in the likelihood function, given information about the model parameters. This open method, in some sense, might decrease the particles degeneracy.
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List of Abbreviations and Acronyms

a.s. - almost surely

\textit{e.g.} - \textit{exempli gratia} (for the sake of example)

\textit{i.e.} - \textit{id est} (that is)

BPF - branching particle filter

CCD - charge-coupled device

CV - coefficient of variation

DSP - digital signal processor

EKF - extended Kalman filter

ESS - effective sample size

FKK - Fujisaki-Kallianpur-Kunita

FPGA - field-programmable gate array

FS - Fisk-Stratonovich

GMCF - generalized Monte Carlo filter

HMM - hidden Markov model

i.i.d. - independent and identically distributed

INS - inertial navigation system

IPF - interacting particle filter

KS - Kallianpur-Striebel

MBPF - modified branching particle filter

MC - Monte Carlo
<table>
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<th>Abbreviation</th>
<th>Description</th>
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<tr>
<td>MV(n)</td>
<td>minimal variance property with n elements</td>
</tr>
<tr>
<td>ODE</td>
<td>ordinary differential equation</td>
</tr>
<tr>
<td>PDE</td>
<td>partial differential equation</td>
</tr>
<tr>
<td>pdf</td>
<td>probability density function</td>
</tr>
<tr>
<td>RDE</td>
<td>random differential equation</td>
</tr>
<tr>
<td>RMSE</td>
<td>square root of the mean square error</td>
</tr>
<tr>
<td>SDE</td>
<td>stochastic differential equation</td>
</tr>
<tr>
<td>SIR</td>
<td>sequential importance resampling</td>
</tr>
<tr>
<td>SIRE</td>
<td>sampling importance restoration</td>
</tr>
<tr>
<td>SIS</td>
<td>sequential importance sampling</td>
</tr>
<tr>
<td>SMC</td>
<td>sequential Monte Carlo</td>
</tr>
<tr>
<td>SNR</td>
<td>signal to noise ratio</td>
</tr>
<tr>
<td>SPDE</td>
<td>stochastic partial differential equation</td>
</tr>
<tr>
<td>SR</td>
<td>systematic resampling</td>
</tr>
<tr>
<td>TBBA</td>
<td>tree-based branching algorithm</td>
</tr>
<tr>
<td>r.v.</td>
<td>random variable</td>
</tr>
<tr>
<td>w.p.1</td>
<td>with probability 1</td>
</tr>
<tr>
<td>w.r.t.</td>
<td>with respect to</td>
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List of Symbols

\(\triangleq\) - is defined as
\(\approx\) - is approximated as
\(\gg\) - much greater than
\((x, y)\) - scalar product between the variables \(x\) and \(y\)
\([x]\) - integer part of \(x \in \mathbb{R}\)
\(\{x\}\) - decimal part of \(x \in \mathbb{R}\), i.e \(x \triangleq x - [x]\)
\(\wedge\) - minimum
\(\vee\) - maximum
1. i. m. - limit in the mean
\(B_t\) - standard scalar Brownian motion
\(W_t\) - standard D-dimensional Brownian motion
\(V_t\) - standard M-dimensional Brownian motion
\(A\) - dimension of the control signal vector
\(D\) - dimension of the state space vector
\(G(t, x)\) - diffusion matrix
\(M\) - dimension of the observation vector
\(N\) - number of discretized intervals
\(P\) - number of samples
\(\{X_t\}\) - signal process at time \(t\)
\(\{X_k^\Delta\}\) or \(\{X_{\tau_k}^\Delta\}\) - discretized signal process at sampled time \(\tau_k\)
\{Y\} - observation process

\( Y \) - a path-valued r.v. of the process \{Y\}

\{Z\} - alternative process of the robust filtering equation

\( u_i \) - \( i \)th control signal, \( i = 1, 2, \ldots, A \)

\( x_i \) - \( i \)th state signal, \( i = 1, 2, \ldots, D \)

\( z_i \) - \( i \)th observed signal, \( i = 1, 2, \ldots, M \)

\( \mathcal{B}^D \) - Borel set in \( \mathbb{R}^D \)

\( \mathcal{F} \) - \( \sigma \)-algebra

\( \mathcal{L}^2_T \) - complete normed linear space

\( \mathcal{N}(\mu, Q) \) - Gaussian r.v. with mean \( \mu \) and covariance \( Q \)

\( \mathcal{O}(x) \) - order of the variable \( x \) (big O notation)

\( \mathcal{S}^2_T \) - subset of all step functions in \( \mathcal{L}^2_T \)

\( \mathcal{U}[x_i, x_f] \) - uniform r.v. for the interval \([x_i, x_f]\)

\( \mathbb{P} \) - probability measure

\( \mathbb{P}^\Delta \) - discretized probability measure

\( \gamma \) - order coefficient of the discretization scheme

\( \delta_0 \) - noise bandwidth

\( \xi \) - forcing process denoting disturbance applied to a plant and its actuators

\( \zeta \) - forcing process denoting disturbance applied to sensors

\( \Delta \) - discretization interval

\( \Omega \) - space of elementary events

\( tr(Q) \) - trace of the square matrix \( Q \)

\( Q^T \) - the transposed of the matrix \( Q \)

\( Q^* \) - the conjugated of the matrix \( Q \)

\( C(E) \) - set of continuous functions on space \( E \)

\( C_b(E) \) - space of continuous bounded functions on space \( E \)

\( C_0(E) \) - space of continuous functions which vanishes at infinity
1 Introduction

1.1 The Control Problem

The study of the Control Problem measures men’s capability to control the nature according to determined constraints. Of course, due to the complexity level of this problem, the attempt to lead the nature to perform with respect to men’s will involves the integration of many areas in the mathematical sciences, making this topic incredibly fascinating.

From the technical point of view, the Control Problem consists of determining a set of mathematical solutions which has influence over a given physical system - a delimited part of the Universe (Faleiros; Yoneyama, 2002) - whose behavior does satisfy well-known performance specifications. This definition shows the necessity to divide the solution of the problem in the following branches: 1-) the mathematical modelling approach of the physical system, 2-) the mathematical transcription of verbal indices describing the performance specifications, and 3-) the closed-loop control strategy. Actually, all these branches are extremely connected.

A physical system can be decomposed into three blocks: 1-) plant, 2-) actuators or manipulated inputs, and 3-) sensors or measurable outputs. The plant refers to the dynamical system to be controlled, e.g. a missile. The actuators are transducers which provide excitation signals in order for the plant to use the control signals $u_1, u_2, \ldots, u_A$, $A \in \mathbb{N}$, supplied by the controller - the entity which implements a given control law designed to make the physical system perform according to a set of specifications. The controller receives the output signals $z_1, z_2, \ldots, z_M$, $M \in \mathbb{N}$, of the physical system from
the sensors - transducers which read the plant’s measurable variables. The type and the
quantity of actuators and sensors in a physical system depend on the nature of the plant
and on the control strategy adopted. In a visually guided missile during an engagement
task, for instance, the seeker’s charge-coupled device (CCD) and the inertial navigation
system (INS) provide the relative position of the target to the autopilot controlling the

Non-manipulated inputs of a physical system, also called *disturbance inputs* or *external
noise* (VAN KAMPEN, 1981), are unpredictable components of the *environment* - the part
of the Universe which is not included in the physical system - that affect the output
adversely. For the sake of simplicity, the plant and the actuators are affected by inputs
denominated \( \xi \) from now on, while the sensors are influenced by \( \zeta \). These inputs may or
may not have causality relations between each other.

The first step to solve the Control Problem is to define a mathematical representation
for the physical system to be controlled. This task, the *mathematical modelling* aims at
describing the most relevant properties of this system via a finite set of differential or
algebraic-differential equations based on physical laws and/or empirical representations
of phenomena. In this step, verbal indices like *system accuracy* and *velocity of response*,
*level of rejection to disturbances*, *minimum time* and/or *energy*, and *simplicity* must be
adequately rewritten according to the chosen mathematical representation.

The type of equation involved in the modelling has strong influence on the extent to
which analytical or even numerical methods can be used to solve the Control Problem.
The *classical control* (NYQUIST, 1932)-(BODE, 1945), for instance, was developed based on
frequency-domain analysis using the concept of *external models*, or transfer functions. A
dynamical system can be completely described by a set of ordinary differential equations
(ODEs) which characterizes at any moment the so-called *states variables*, e.g. velocities
and positions. The *modern* state-space methods (KALMAN, 1960)-(PONTRYAGIN et
al., 1964), or *internal models*, were introduced based on the time-domain analysis using
information from these variables.
CHAPTER 1. INTRODUCTION

The complexity level related to the design of a control law depends exclusively on the required set of specifications and on the mathematical details that are inherent to the dynamic provided by the physical system of interest. One important point in designing the control strategy is to know about the availability of manipulated inputs and measurable outputs of the physical system. Many times these inputs and outputs are restricted because there must be a trade-off between the system performance and the financial costs related to the given engineering project.

For a deterministic system, i.e. a physical system whose main aspects are supposed to be completely described by a set of ODEs, the problem of knowing the values of internal variables that cannot be directly measured is solved by the so-called observer [Luenberger, 1971].

However, when uncertainties are considered in the internal states of the model and/or in the measuring process, the observer does not provide optimal estimates and random variables must now be introduced into the plant modelling. Systems modelled according to this kind of approach are named stochastic ones. Here the so-called filter is the tool used to rebuild with a known uncertainty level the non-accessible states of the physical system, once the mathematical model is supposed to provide accurate information about the noise structure. Filters can also be applied aiming only at decreasing the system uncertainties of the accessible states, as presented in approaches of the classical statistics [Papoulis, 1965] - [Peebles, 1993]. The performance of a control strategy can be improved as long as the measured variables have the signal-to-noise ratio improved.

Figure [1.1] represents the basic block diagram describing the subsystems composing the general solution of the Control Problem. Given the mathematical model describing the main characteristics of the physical system, the objective is to design a controller able to make the plant perform in accordance with pre-defined specifications. Particularly, if the physical system is represented by internal states, these are taken to the external world by using sensors with particular dynamics. Obviously, these sensors are not used in all internal states because some modelled variables may not have any physical significance.
The controller block is out of the scope of this thesis. The Control Theory for deterministic systems can be understood in the books of Franklin, Powell and Emami-Naeini (2002) for a basic introduction, and Sontag (1998) and Khalil (1996) for a deeply mathematical study. For stochastic systems the control approach is described in the seminal books of Astrom (1970) and Jazwinski (1970), and after in Maybeck (1982) and Goodwin and Sin (1984).

This thesis is focused on the investigation of the filtering block. For this, the study of stochastic modelling approaches is of great importance, once the filtering performance is better as the parametric modelling is done in precise and realistic ways. Furthermore, this thesis is concerned with the problem of nonlinear filtering, specially in procedures for implementing numerical approaches in real-time applications. Then, the use of computers is of paramount importance. The filtering problem can be addressed using the following steps: 1-) the solution of the stochastic modelling of dynamical systems, 2-) the solution of the nonlinear filtering in the continuous and discrete-time approaches, and finally 3-) the
implementation of the mathematical solutions by using huge computational algorithms, such as the Monte Carlo-based ones.

In its general scope this thesis presents the nonlinear filtering problem motivated by applications in the Control Systems field, specifically aerospace systems. This text discusses since the important aspects in modelling dynamic systems via stochastic variables up to the computational implementation of Monte Carlo-based nonlinear filters, approaching also the theoretical aspects and interpretative questions of the nonlinear filtering theory.

This work also intends to clarify theoretical aspects which affect directly the implementation of the nonlinear filters in practical applications. In this way, the author is concerned, many times, in rewriting the mathematical notation in a friendly and unified format that can be easily understood in many areas of engineering and mathematical sciences.

1.2 Motivations

Many mathematical solutions for the Control Problem involve the integration of multidimensional equations and/or their parametric optimization with respect to a given set of performance indices. Very complex solutions from the analytical or even numerical viewpoints can be obtained when realistic results are required in the design of the control law, once a physical system can be modelled considering non-linearities, specific operational modes, internal and/or external perturbations, among other factors (GARCÍA, 1997). By focusing now on Control Problems of aerospace systems, which naturally claim for real-time processing, it is really important that the chosen mathematical approach can obtain solutions that are simple, efficient and easy to be implemented in fast electronic systems. These needs suggest a trade-off between performance and ways of implementing the solution under restrictions of computational workloads and financial costs.

In the general case, assuming that a physical system is modelled according to the modern Control notation, the variables, or states, whose quantity are associated to the
dynamic system order, are related among themselves by nonlinear functions with additive disturbances acting as random forces, the so-called extrinsic disturbances. Each state can contain a set of parameters empirically obtained, what implies the existence of modelled additive uncertainties too, the so-called intrinsic disturbances. These uncertainties are introduced because systems are too complex to be described in detail, or simply because a detailed description is too difficult to handle [PETERS, 2000]. The stochastic aspect, thus, is introduced in order to model the incomplete knowledge.

The level of generality for the so-called closed estimate problems presented by Kalman (1960), Benes (1981) and Brockett (1981) makes the obtention of analytical forms via finite-dimensional filters practically impossible. In this way, in order to achieve the required objectives, it is necessary the associated solutions, now represented by an integral form and/or multi-dimensional equations, to be approximated via numerical methods that, depending on the class of problems [FERZIGER 1998], can be extremely complex from the computational viewpoint and not sufficiently accurate and robust [HIGHAM, 2002].

Usually, to make the solutions feasible for real-time applications, a standard approach is considered. Such approach consists of making simplifications of the models, or even of the analytical approximations, what can be difficult mainly in a field with hard performance specifications, as in aerospace systems. It is important, therefore, for a great improvement of quality in the development of solutions for Control Problems, the search for numerical methodologies that, beyond giving results as expected by the theoretical approaches, are implementable in fast electronic systems preserving remarkable characteristics of the model without degrading the performance.

With the recent availability of affordable electronic devices with high capacity of processing, approaches based on numerical simulation with complexity of real problems become a reality for the solution of the Control Problem applied to embedded systems. Monte Carlo (MC) algorithms [HAMMERSLEY; HANSCOMB, 1964] are an attractive alternative thanks to their parallelization facility [MASCAGNI, 1991], since they can be
extremely precise, robust and sufficiently flexible for different forms of implementation.

In the decade of 1940, John von Neumann presented a formal representation of MC methods and established the mathematical basis for the probability density functions (pdf), inverse cumulative distribution functions and pseudo-random numerical generators. Stanislaw Ulam, who recognized the potential of the just invented electronic computer in the implementation of the new approach, developed jointly with Nicholas Metropolis algorithms exploring forms of transforming non-random problems in random forms which could facilitate their solution by statistical sampling [METROPOLIS; ULAM, 1949]. The applications of the method grew mainly during the Manhattan Project [RHODES, 1986] with the designing of nuclear reactors shielding. In the end of the years of 1960, MC methods were tested in a variety of fields of engineering, however even the simple problems still had computation limitations due to the low computational processing capacity of those times. Even in the 1970’s and 1980’s, many complex problems could not be solved, mainly for real-time applications.

MC methods, as comprehended nowadays, denote any statistic sample technique employed to approximate solutions for quantitative problems. The most interesting thing is that they may be applied even to problems without probabilistic contents, but with self probabilistic structure, as in the approximations of solutions of partial differential equations (PDEs) [KUSHNER, 1977]-(KUSHNER; DUPUIS, 2001)-(KUSHNER; YIN, 2003)-(MILSTEIN; TRETYAKOV, 2004) and integrals related to conditional expectations [GAMerman, 1997]-(GILKS; RICHARDSON; SPIEGELHALTER, 1996)-(ROBERT; CASELLA, 2004). Among all numerical methods which make $P$ pointwise computation in a $D$-dimensional space to obtain an approximated solution, MC method has an absolute error of the estimate which decreases with $P^{-1/2}$ while the other methods, without exploring completely the search space, have error which decreases with $P^{-1/D}$ in the best hypothesis [RUBENstein, 1981].
1.2.1 MC Methods to Solve Problems in Systems

Traditionally MC methods have been developed by several branches in Physics, mainly in areas related to the simulation of atom chain behavior (Li, 2001). In the end of the 1970’s and during the 1980’s, methods based on the use of probabilities to approximate PDEs (Kushner, 1977) representing stochastic models (Freidlin, 1985) emerged as a motivation to use MC simulation applied to important problems in Control, as the nonlinear filtering one (Kushner, 1979)-(Davis, 1981a)-(Talay, 1982; Talay, 1984)-(Le Gland, 1984)-(Pardoux; Talay, 1989). However, these methodologies could not be implemented in real-time applications, once in that time there were no electronic devices capable to satisfy the trade-off between performance and financial costs. Then the research in this area almost stopped and only restarted in the middle of the 1980’s with important results coming from statistics (Gilks; Richardson; Spiegelhalter, 1996)-(Gamerman, 1997)-(Doucet, 1998) and from applied mathematics (Del Moral, 1996b)-(Del Moral, 1996a)-(Crisan, 1996).

In the last 10 years, thanks to the discovery of more efficient MC algorithms (Earnhead, 1998)-(Doucet; Freitas; Gordon, 2001b)-(Robert; Casella, 2004)-(Milstein; Tretyakov, 2004) and also to the facilities of their implementation in programmable electronic devices (Bolic, 2004), studies of MC methods for Control Problems have been developed with promising results for detection and target tracking (Ristic; Arulampalam; Gordon, 2004)-(Pavlov, 2004), vehicle navigation and guidance (Bergman, 1999) and failure identification (Andrieu et al., 2004). A general discussion on the applicability of MC methods to problems in automatic control was presented by Vidyasagar (1998).

1.2.2 Nonlinear Filtering

The study of the nonlinear filtering problem has been initiated by Stratonovich (1960b) and Kushner (1967a). A generalization and a proof using martingale theory is due to
Fujisaki, Kallianpur and Kunita (1972). An approach leading to dynamical equations for a non-normalized conditional density was developed by Kallianpur and Striebel (1968), Mortensen (1966), Zakai (1969) and Partloux (1979). None of these filtering formulas are directly implementable, since all are “infinite-dimensional”, i.e. describe the time evolution of conditional distribution or density functions in the form of measure-valued or stochastic partial differential equations (SPDEs). Benes (1981) discovered a class of non-linear systems for which the conditional density admits a finite-dimensional parameterization, and this has led to extensive research on characterizing such systems and exploring the connection, uncovered in Brockett and Clark (1980), between non-linear filtering and certain Lie algebras of differential operators.

Recursive formulas updating the least-squares estimate were obtained by Stratonovich (1960a), Stratonovich (1960b), and after by Kushner (1964). However, it was found that in general these formulae involve all the conditional moments, so that an infinite set of simultaneous equations or, equivalently, a PDE for the conditional probability density or the conditional characteristic functional is necessary. Satisfactory approaches to solving directly, even approximately, the SPDE of Zakai (1969) seem to be at hands with the computational approaches developed by Bensoussan, Glowinski and Rascanu (1990), Le Gland (1992) and Bensoussan, Glowinski and Rascanu (1992), Kushner (1977), Sun and Glowinski (1993) and Ito and Rozovskii (2000). In these approaches the PDE associated to the solution of the nonlinear filtering problem is solved.

Recently contrasting approaches on numerical analysis have been developed in order to approximate the solution of the standard filtering problem. Engineers often like to use the extended Kalman filter (EKF) (Anderson; Moore 1979) due to its facility of implementation and satisfactory performance. Approximations by finite-dimensional nonlinear filters have been used (Benes 1981)–(Daum 1988) as an alternative for a restrict class of dynamic systems.

There are essentially two different approaches to the nonlinear filtering problem. The first is based on the important idea of innovations processes, originally introduced by
Hendrik W. Bode and Claude E. Shannon in the context of Wiener filtering problems. This approach reaches its culmination in the seminal paper of Fujisaki, Kallianpur and Kunita (1972). A detailed account of this approach is available in the books of Lipster and Shiryaev (1977) and Kallianpur (1980). The second approach can be traced back to the works of Mortensen (1966), Duncan (1964) and the important paper of Zakai (1969). In this approach the attention is focused on the unnormalized conditional density equation, which is a bilinear PDE, and it derives its inspiration from function space integration as originally introduced by Kac (1951). Mathematically, this last view is closely connected to the path integral formulation of Quantum Physics (Feynman; Hibbs 1965).

The nonlinear filtering problem can also be divided in the continuous and discrete-time approaches. The former solves the problem when the physical system is modelled according to continuous-time state space and observation. The last uses a discretized physical system. From the practical viewpoint, even for the continuous-time case, given the hardness to obtain a closed analytical solution, there is the need to discretize the representation equations in some part of the solution. In the discrete-time case, for instance, the state and observation equations are already discretized and the Bayesian approach is naturally developed according to the traditional statistics. For the continuous-time case, however, the system equations are kept continuous in order to deduce iterative equations which use information about the noise parameterization, such as Zakai equation. But to solve the problem it is necessary to discretize in some way the equation. Furthermore, both approaches use different concepts of weights representing the likelihood function sampled by MC methods of integration.

Given \((\Omega, \mathcal{F}, P)\), a probability space, the mathematical rigorous model for the continuous-time nonlinear filtering problem consists of estimating a time-homogeneous Markov process \(\{X_t\}_{t \geq 0}\) with known law, given the process \(\{Y_t\}_{t \geq 0}\) defined by

\[
Y_t = \int_0^t h(s, X_s) \, ds + V_t, \quad 0 \leq t \leq T,
\]
where the signal process, \( X \), takes values in \( \mathbb{R}^D \) and the observation one, \( Y \), in \( \mathbb{R}^M \), with \( D, M \geq 1 \). The process \( \{ V_t \}_{t \geq 0} \) is a standard \( M \)-dimensional Brownian motion, \( h : \mathbb{R}^+ \times \mathbb{R}^D \to \mathbb{R}^M \) is a bounded continuous function, and \( T \) is a given final time. The initialization conditions are assumed to be \( Y_0 = 0 \), whereas \( X_0 \) is a random variable (r.v.) with law \( \pi_0 \).

Let \( \mathcal{F}_0^t \) be the filtration generated by \( Y \) up to time \( t \) and \( \varphi : \mathbb{R}^D \to \mathbb{R}^M \) be a bounded continuous function. The classical filtering problem can be summarized as finding the conditional distribution of \( X_t \) with respect to \( \mathcal{F}_0^t \), i.e.

\[
\pi_t(\varphi) = \int_{\mathbb{R}^D} \varphi(x) \pi_t(dx) = \mathbb{E}[\varphi(X_t)|\mathcal{F}_0^t],
\]

(1.1)

where \( \pi_t(\varphi) \), the so-called normalized representation, is known to satisfy the measure-valued SDE named Fujisaki-Kallianpur-Kunita (FKK) [Fujisaki; Kallianpur; Kunita, 1972] and Kushner-Stratonovich (Kushner, 1967a). Another representation of equation (1.1) is given by the so-called Kallianpur-Striebel (KS) formula [Kallianpur; Striebel, 1968]

\[
\pi_t(\varphi) = \frac{\mathbb{E}[\varphi(X_t)\Lambda_t|\mathcal{F}_0^t]}{\mathbb{E}[\Lambda_t|\mathcal{F}_0^t]} = \frac{\sigma_t(\varphi)}{\sigma_t(1)},
\]

(1.2)

where \( \mathbb{E} \) is the conditional expectation with respect to the measure defined by \( \Lambda_t \), and \( \sigma_t(\varphi) \), the unnormalized representation, satisfies Zakai (Zakai, 1969) and the pathwise (Clark 1978)-(Davis, 1981b)-(Par dor, 1981) equations.

MC methods have presented promising results in approximating numerically solutions of the nonlinear filtering problem. The basic idea is to represent the likelihood function associating weights to the realizations/samples obtained from the probabilistic model. The estimate is then obtained by using the traditional MC integration, where the properties of convergence are well-known (Hammersley; Handscomb, 1964). The need to implement these filters in practical problems has motivated the study of approaches which select the likelihood weights aiming at improving the convergence rate of the obtained estimates.
Specifically, these new methods are called particle filters. The realizations, also denoted as particles, are multiplied or removed from the simulation in accordance with a criterion usually based on the variance of the weights (Liu; Chen 1995). The process of particle selection, usually called resampling, acts directly on the filter performance and can obey several criteria in reference to the weight likelihood, as the systematic resampling (Fearnhead, 1998) and the residual resampling (Bolic, 2004), or even the one associated to the entropy, as the tree-based branching algorithm (TBBA) (Crisan; Lyons, 2002).

The research in the field of particle filters in continuous-time is often classified in two main branches: the study of the approximation of the solution of Zakai equation (Zakai, 1969) using the branching particle filter approach proposed by Crisan and Lyons (1997) and Crisan, Gaines and Lyons (1998), and the study suggested by Davis (1981b) via the pathwise theory (Clark 1978)-(Davis, 1981a)-(Pardoux, 1981). In the first case, performance studies for several forms of implementation of the filters were tested by Jacob and Yoneyama (2004). For the second one, it was proposed a new methodology of implementation by using the simulation of SDEs (Jacob, Hemerly, Yoneyama, 2004)-(Jacob, Yoneyama, Hemerly, 2005), once that originally the numerical solution was implemented using Markov chains (Kushner, 1977)-(Kushner, Dupuis, 2001) simulating PDEs.

About the approach of discrete-time filters, the studies are concentrated in the search for an alternative scheme to replace the usual methodology of resampling of particles for something equivalent, but allowing the parallelization in these architectures. The preliminary development of the restoration method by Jacob and Yoneyama (2005a) appeared as a promising but still opened solution.

### 1.2.3 The Monte Carlo Integration

For simplicity and notational convenience, the presentation given herein assumes integration over the total range of the Euclidean space while the general theory deals with
more abstract spaces in a measure theoretic fashion. Numerical integration deals with the problem of numerically evaluating general integrals given by

\[ I = \int_{\mathbb{R}^D} g(x) \, dx. \]  \hspace{1cm} (1.3)

MC methods for numerical integration regard problems on the form

\[ \hat{I} = \int_{\mathbb{R}^D} \varphi(x)q(x) \, dx, \]  \hspace{1cm} (1.4)

where \( q(x) \) is a positive function that integrates to unity, i.e.

\[ q(x) \geq 0, \quad \int_{\mathbb{R}^D} q(x) \, dx = 1. \]

Most problems on the form (1.3) can be transformed into an integral for MC evaluation through a suitable factorization of the integrand \( g(x) = \varphi(x)q(x) \). The assumptions on the factor \( q(x) \) impose a natural interpretation of \( q(x) \) as a pdf.

MC methods rely on the assumption that it is possible to draw \( P \gg 1 \) samples \( \{x_i\}_{i=1}^P \) distributed according to the probability density \( q(x) \). MC estimate of the integral (1.4) is formed by taking the average over the set of samples

\[ \varphi_P = \frac{1}{P} \sum_{i=1}^P \varphi(x_i). \]  \hspace{1cm} (1.5)

If the samples in the set \( \{x_i\}_{i=1}^P \) are independent, \( \varphi_P \) will be an unbiased estimate and will almost surely (a.s.) converge to \( \hat{I} \), i.e.

\[ \mathbb{P} \left( \lim_{P \to \infty} \varphi_P = \hat{I} \right) = 1 \]  \hspace{1cm} (1.6)

by the strong law of large numbers. Moreover, if the variance of \( \varphi(x) \),

\[ \sigma^2 \triangleq \int_{\mathbb{R}^D} (\varphi(x) - \hat{I})^2 q(x) \, dx = \int_{\mathbb{R}^D} \varphi^2(x) q(x) \, dx - \hat{I}^2 \]  \hspace{1cm} (1.7)
is finite, then the central limit theorem yields convergence in distribution of the error

\[
\lim_{P \to \infty} \sqrt{P} (\varphi_P - \hat{I}) \sim \mathcal{N}(0, \sigma^2),
\]

where \( \mathcal{N}(\mu, Q) \) means Gaussian distribution with mean \( \mu \) and covariance \( Q \).

**Remark 1.2.1.** Even in cases where the samples in the set \( \{x_i\}_{i=1}^P \) are dependent, it is also possible to obtain a law of large numbers and a central limit theorem under weak assumptions [Hammersley; Handcomb 1964].

The convergence results given by equations (1.6) and (1.8) are asymptotic. This means that as \( P \to \infty \) the error of the approximation will tend to zero. With support from this asymptotic result it is usually assumed that a large but finite \( P \) will lead to a small error. In practical applications the number of samples might be very large for a given error bound. There are, however, two main advantages of MC integration compared to straightforward numerical integration. The traditional methods generally suffer from intractable demands for computational resources and implementational complexity when applied in high dimensional spaces. Equation (1.8), on the other hand, yields that the error \( \varepsilon = \varphi_P - \hat{I} \) of MC estimate is of the order

\[
\varepsilon = O(P^{-1/2}),
\]

independently of state dimension.

Moreover, while the numerical integration methods require the user to define a grid over the integration area that naturally is dependent on the integrand, the estimate given by equation (1.5) is obtained using the same technique for any function \( \varphi(x) \). One should note though, that even if the error will tend to zero asymptotically in \( P \) at a rate that is independent of the state dimension, the constant factor hidden behind equation (1.9) usually will depend on the state dimension. A given bound on the error \( \varepsilon \) will often demand more samples in high dimensional problems than in low dimensional ones.
The origin of the positive effects of MC integration can be found in the fact that for this method, the set of samples \( \{x_i\}_{i=1}^P \) is automatically chosen in the parts of the state space that are important for the integration result. Since the samples are chosen according to the density \( q(x) \), which is a factor of the integrand, one can conclude that the effectiveness of the method depends on how the factorization \( g(x) = \varphi(x)q(x) \) is performed. The more informative, or varying, \( q(x) \) is compared to \( \varphi(x) \), the better will this automatic choice of sample locations be. This claim is verified by equation (1.7) since the relative smoothness of \( \varphi(x) \) compared to \( q(x) \) determines the size of \( \sigma^2 \). This variance directly affects the size of the error through the limit defined by (1.8). The reduction of the variance and thereby the reduction of the estimation error can be achieved in several ways (Davis; Rabinowitz [1984]). The techniques for variance reduction usually rely on approximation of the integrand by functions that may be handled analytically. Therefore, these methods may not be applicable when the dimension is high or the integrand has no closed form analytical expression.

The numerical integration methods generally approximate the integral by a summation over a grid of regular discretization manually chosen at the support set of the integrand. MC methods utilize the fact that, under the assumption that it is possible to generate \( P \) samples from a density given as a factor of the integrand, an adaptive grid well suited for integral approximation is, more or less, automatically obtained. This, in a sense, is the way these methods beat the curse of dimensionality and is the core difference between straightforward numerical integration and MC integration methods.

Actually, the efficiency of MC integration can be raised beyond the error defined by (1.9), once an error which asymptotically is inversely proportional to \( P \) is achieved. This is obtained by using, not independent random variables in the set \( \{x_i\}_{i=1}^P \), but choosing each new sample \( x_i \) sufficiently far away from the former ones, so that any potential clustering of the samples in the sum presented by equation (1.5) is alleviated. The means for obtaining such pseudo random variables rely on the theory of equidistant sequences (Davis; Rabinowitz [1984]), and these methods are sometimes named quasi MC meth-
ods (NIEDERREITER, 1978). However, generating equidistant sequences with distribution according to an arbitrary density $q(x)$ is in general rather difficult (FEARNHEAD, 1998).

Simulation based on MC integration methods promises general solutions to complex and high dimensional problems of numerical integration and optimization. These methods have the main advantage over the classical numerical integration because the relative error is of order $O(2)$, not explicitly depending on the state dimension. Still, MC methods remain computer-intensive and generally put high demands on both the computational resources and the available memory size.

### 1.2.4 MC Methods for Real-Time Systems

Applications in aerospace systems naturally require the use of real-time processing. In this field, the solutions obtained for Decision and Control Problems are usually put in the form of mathematical representations, as multi-dimensional integrals. Their feasible implementations are, of course, numerical approximations.

For embedded systems, given the trade-off between the computational capacity and the cost of the programmable electronic devices, often the sampled signals have low precision and the computed variables have restrictions due to the mantissa. These data must be considered for the performance validation of an algorithm being tested, mainly when theoretical results guarantee it to work. For example, results with respect to the convergence rate of an approximation may be affected or even become unusable for a specific application. Another important point is the parameterization of these algorithms in accordance with digital signal processors (DSPs) and field-programmable gate arrays (FPGAs) architectures. These problems might make the implementation of theses algorithms a great challenge!

MC methods and their variations are very attractive, once they can, depending on the number of available samples, be extremely precise and robust. Thanks to their facility of parallelization and computational distribution (MASCAGNI, 1991), they are very flexible.
with respect to the implementation forms.

In the nonlinear filtering field, there are limitations in the implementation of the particle filters due to the resampling scheme (Bolic, 2004). Many efforts have been done in the search for different approaches that avoid the weight degeneracy (Liu; Chen, 1995), as presented in the papers of Liu, Chen and Logvinenko (2001), Godsill and Clapp (2001) and Jacob and Yoneyama (2005a).

1.3 Main Results

Into its general scope, this thesis presents MC-based solutions for the nonlinear filtering problem motivated by control applications in the aerospace field. The solution of the general problem requires investigations in many scientific subjects, such as stochastic modelling, nonlinear filtering theory, MC schemes for approximating integrals, and computational methodologies applied to real-time systems.

This thesis is focused on two basic aspects of the Control Problem: 1-) the stochastic modelling of physical systems, and 2-) MC-based numerical approximation of the nonlinear filtering problem solution.

In the first topic this thesis concerns about clarifying some issues in the mathematical modelling of continuous-time systems with Brownian motion. The hypothesis that physical systems should be modelled in continuous-time approach is defended, once the main results in Physics provide solutions for dynamic systems via continuous-time differential equations. It was shown, recalling a main result from Wong and Zakai (1965) and Clark (1973), that a physical system is represented by Fisk-Stratonovich (FS) SDE, though Ito approach is better to manipulate the mathematical operations. In other words, the idea is to present the main results in stochastic theory demonstrating the effectiveness of FS approach, instead of the Ito one, in modelling physical systems and in simulating their respective equations using computers. In both cases, it is shown that stochastic integrals based on very small correlation uncertainties, e.g. those ones obtained in physical pro-
cesses and computer-generated random numbers, converge to FS integral in mean-square.

For practical calculations, it is convenient to switch over from FS SDE to the equivalent Ito form. The main advantage of Ito version of SDE is the direct displaying of the characteristics of a given diffusion process by which a system coupled to an environment with extremely rapid fluctuations has to be modelled. Ito counterpart is a useful artifice which allows access to the appropriate moment equations or the Fokker-Planck equation.

These equations in general do not have analytical solutions and their solution properties just can be studied by using numerical methods, such as the well-known Taylor-type approximation schemes of Euler-Maruyama and Milstein \cite{KLOEDEN2:1999}. These schemes exemplify discretization methods which allow the simulation of continuous-time mathematical models in computers. An alternative approach for stochastic modelling well diffused among engineers is to add white noise after discretizing ODEs representing a physical model. This thesis presents examples elucidating and discussing the mathematical conditions in which the alternative modelling approach is equivalent to the discretization of stochastic continuous-time model with Brownian motion, as preliminary discussed by Jacob, Silva and Yoneyama \cite{JACOB2:2006}.

The nonlinear filtering problem is referred to the second and last topic. A unified treatment of the available MC methods solving the nonlinear filtering problem for continuous and discrete-time modelling is presented with sufficient emphasis on basic applications enabling the engineer to use the main results provided by the theory. The need for this topic is twofold. First, although nonlinear estimation theory is well known, its notation is largely scattered in the journal literature. Second, available literature on the continuous nonlinear theory can be inaccessible to engineers uninitiated in measure theory and SDEs. This topic can be branched as follows: 1-) study of the theory of nonlinear filtering problem in continuous and discrete-time approaches, and 2-) investigation of the main aspects of MC-based numerical solutions approximating the integrals representing the conditional expectation of a signal process function given the observations.

The results on continuous-time modelling approach presented in this thesis are from
CHAPTER 1. INTRODUCTION

the classical nonlinear filtering theory, mainly the ones whose representation is developed using unnormalized conditional expectations. This form to present the results facilitates the use of Feynman-Kac \cite{DelMoral2004} to solving nonlinear filtering problem. The theory was represented by Zakai equation \cite{Zakai1969} and the pathwise or robust formulation \cite{Clark1978,Davis1979}. One of the contributions of this thesis is to rewrite and present the robust formulation for a more general case where the observation process contains non-standard Brownian motion, in accordance with preliminary manipulations developed by Pardoux \cite{Pardoux1981} and Souza \cite{Souza1992}. The Bayesian approach was discussed after the presentation of the discrete-time modelling. The difference between both approaches is that in the former the filtering equations are presented explicitly, while the last one uses general representations of pdfs to manipulate the results.

From the practical viewpoint, the continuous-time theory solving the filtering problem can be of great importance when the representations are discretized and then implemented in computers. After rewriting the classical approach using Feynman-Kac notation, it is then possible to implement numerically the conditional expectation integrals using MC methods, as suggested by Talay \cite{Talay1982} and Talay \cite{Talay1984} that discretized the Girsanov exponential, and by Davis and Wellings \cite{Davis1980} for the robust representation.

The use of MC methods to approximate the solutions enabled the use of resampling schemes attempting to improve the convergence rate of the estimates. The Girsanov exponential-based approach has been deeply studied using particle filtering methods as the branching particle filter (BPF) \cite{CRISAN1997,CRISAN2003} and the interacting particle filter (IPF) \cite{DelMoral1996b,DelMoral1996a}. Specifically for BPF, this thesis presents a study stating the difficulties, e.g. the control of the number of particles, to implement the kernel idea of the method in practical situations, as described by Jacob and Yoneyama \cite{Jacob2004}.

The need for robust representation arises when the filtering framework is used to model and solve real-life problems, once the observation model chosen for the real observation process may not be a perfect one. In this way, the search for a particular version of
the conditional expectation which has nice properties, e.g. the representation is locally Lipschitz given the modelling imperfections (Clark; Crisan 2005), is of paramount importance for applications of the nonlinear filtering theory in practice.

Applying a variable transformation in the Girsanov exponential, Clark (1978) and Davis (1979) showed that the solution of the nonlinear filtering formula can be obtained separately for each sample path of the observation model. The numerical solution suggested by Davis (1981a) to simulating trajectories used ideas from Kusleen (1977) which approximated PDEs by using a Markov chain. It is now known that the simulation of trajectories using discretized SDEs has better accuracy and less computational effort than those required by the Markov chain case (Kloeden; Platen 1999). This thesis replaced the traditional method by the simulation of trajectories via the Milstein scheme. Investigations were made in order to compare the estimate performance with the results of BPF and the extended Kalman filter (EKF) as well, according to the ideas developed by Jacob, Hemerly and Yoneyama (2004) and Jacob, Yoneyama and Hemerly (2005).

The study of the numerical solutions of the pathwise filter brings some interesting results still not reported in the classical papers of Davis (1981a) and O’Loghlen and Wright (1982), and in the doctoral thesis of Souza (1992): the numerical estimates can diverge when the signal-to-noise ratio is low, an important topic of this thesis. Preliminary results presented by Jacob and Yoneyama (2005a) showed that the variance of the weights increased dramatically when the signal simulated by a stable Ornstein-Uhlenbeck process is near zero and is disturbed by a significative noise amplitude. The first idea to explain the causes of the problem was that the unfavorable noise amplitude might make the filter perform badly. The solution suggested was to run the particle resampling (Liu; Chen, 1995) at time instants determined by the effective sample size (Doucet, 1998).

The study of this divergence problem in the pathwise filter is extended in this thesis. The problem has occurred because it is necessary to simulate an alternative Ornstein-Uhlenbeck process with drift depending on the observations. Particularly for the case where the signal and the observation processes are close to zero, this drift term can make
the alternative SDE unstable, what can accumulate significant errors to the numerical simulation of the solution.

To show the enormous potential of the use of particle filters in the aerospace field, it is of great importance to verify the possibility to implement these methods in real-time systems. Although MC methods are really attractive due to their facility of parallelization and computational distribution (MASCAGNI, 1991), the main drawback is the well-known degeneracy phenomenon of the particles. As time goes by, it is necessary a greater number of particles to keep a stabilized accuracy of the estimates. This problem is solved by using the resampling scheme, a stage in MC filtering which makes difficult the parallelization of the algorithm (BOLIC, 2004). The restoration method proposed in this thesis is presented according to the preliminary results described by Jacob and Yoneyama (2005a) using the Bayesian approach for discretized models. This open alternative method was developed to move the particles towards higher regions in the likelihood function, given information about the model parameters. This method, in some sense, decreases the weight degeneracy, but seems to introduce some bias to the estimates.

Next sections present each one of the contributions of this doctoral thesis in the context of the specific problems solved.

1.3.1 Stochastic Modelling

The problem of designing tractable mathematical models which adequately describe the behavior of a physical system is of great importance. For very complex nature systems, e.g. ones difficult to be completely represented by a finite set of equations based on physical laws or mathematical descriptions of phenomena, simplifications should be made in the state-space model. This lack of knowledge, along with possible natural statistical fluctuations in the states, brings uncertainties which can be modelled by adding random forces whose finite-dimensional distributions are supposed to be known. Continuous-time physical systems should be entirely modelled in the continuous-time paradigm, in
contrast to the well-performed procedure which first discretizes the deterministic state-space model and then adds white noise to the states. This alternative approach, though easy to be comprehended, is conceptually incorrect and not general from the stochastic theory viewpoint (Jacob; Silva; Yoneyama 2006). For continuous-time mathematical models, the inclusion of random effects with nondifferentiable sample paths, as Brownian motion (Wax, 1954), leads the model to different interpretations due to the nature of the stochastic integral involved.

The main behavior of a plant can be described when an ODE has: 1-) random coefficients, 2-) a random initial value, and/or 3-) is forced by a stochastic process with differentiable sample paths (Sussmann 1978). Equations of this type are often called random differential equations (RDEs) and are given by

$$\dot{X} = \frac{dX}{dt} = f(t, X_t) + \sigma(t, X_t) \xi_t, \quad X_0 \sim \pi_0,$$

(1.10)

where $f : [0, T) \times \mathbb{R}^D \to \mathbb{R}^D$ and $\sigma : [0, T) \times \mathbb{R}^D \to \mathbb{R}^D$, and the forcing process $\{\xi_t\}_{t \geq 0}$ has differentiable sample paths. The solution of RDE (1.10) is obtained by ordinary calculus integrating each sample path independently (Olechzyk, 1991).

When the forcing term is a stochastic process with fairly irregular sample paths, e.g. the Brownian motion, the equation is then written formally as stochastic differentials interpreted as integral equations with Ito (Ito, 1944) or FS (Fisk, 1963)-(Stratonovich, 1964) stochastic integrals. They are called SDEs and in general their solutions inherit the nondifferentiability of sample paths from the Brownian motion in the stochastic integrals (Arnold, 1974). SDE’s theory, developed by Ito (1950), Ito (1951), Doob (1953) and Stratonovich (1964), serves just this purpose.

Replacing the forcing process $\{\xi_t\}_{t \geq 0}$ in RDE (1.10) by the standard Brownian motion $\{W_t\}_{t \geq 0}$, the general equation, now a SDE, is given in the differential formulation by

$$dX_t = f(t, X_t) dt + \sigma(t, X_t) dW_t,$$

(1.11)
or by the equivalent integral formulation

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s) \, ds + \int_{t_0}^{t} \sigma(s, X_s) \, dW_s.$$  \hfill (1.12)

The first integral is defined in the ordinary sense for each sample function and the integrand has the so-called *spurious drift* defined as

$$f(t, x) \triangleq f(t, x) - \frac{1}{2} \sigma(t, x) \frac{\partial \sigma}{\partial x}(t, x).$$  \hfill (1.13)

As the Brownian motion has unbounded variation, the latter integral in equation (1.12) is defined in Itô sense. Assuming that $X_{t_0}$ is a given square-integrable r.v. independent on $\{W_t\}_{t \geq 0}$ and that the functions $f$ and $\sigma$ are globally Lipschitz, this SDE has only one possible solution  \cite{lipster1977}.

In contrast, when the stochastic integral in equation (1.12) is interpreted in FS sense, the given SDE requires a new mathematical notation defined by

$$dX_t = f(t, X_t) \, dt + \sigma(t, X_t) \circ dW_t,$$ \hfill (1.14)

or equivalently by

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s) \, ds + \int_{t_0}^{t} \sigma(s, X_s) \, dW_s.$$ \hfill (1.15)

This thesis recalls an important result from the 1960’s and 1970’s. Given a partition $(\tau)_\Delta$ over $[t_0, t]$ with sample time $\Delta > 0$, the idea is to replace the forcing process $\xi_t$ in RDE (1.10) by a polygonal approximation $W^\Delta_t$ of the standard Brownian motion $W_t$. \cite{wong1965, wong1971, clark1966} and \cite{clark1973} showed that the solutions of RDE (1.10) driven by $W^\Delta_t$ with short correlation time tends, in the mean square sense, to the solution of FS SDE (1.14) as long as the given correlation time tends to zero, as presented in the next proposition.
**Proposition 1.3.1.** Let \( f : [0, T) \times \mathbb{R}^D \to \mathbb{R}^D \), \( \sigma : [0, T) \times \mathbb{R}^D \to \mathbb{R} \) be of class \( C^1 \) and \( \sigma \partial \sigma / \partial x \) satisfy a uniform Lipschitz condition in \( x \) and be continuous in \( t \). Furthermore, suppose the initial condition has finite fourth order moment. Then

\[
\lim_{\Delta \to 0} X^\Delta_t = X_t, \tag{1.16}
\]

where \( \{X_t\}_{t \in [0, T]} \) is the solution of FS SDE (2.4).

**Proof.** See [Wong and Zakai (1965)] and [Clark (1973) Proposition 1]. \( \square \)

In other words, Proposition 1.3.1 states that for modelling plants and/or simulating SDEs by means of analog or even digital computing devices, FS approach provides the most realistic description of the processes [Pugachev; Sinitsyn (1987)]. Therefore, once that physical systems are affected by band-limited white noise and computer-based random generators run r.v.s with a small but different from zero correlation interval, the recommendation is always to use FS SDEs.

In engineering and the physical sciences many mathematical models of plants are obtained by including random fluctuations in ODEs, which have been deduced from phenomenological or physical laws. An alternative approach for stochastic modelling is to add white noise after discretizing ODEs representing a physical model. The question here is to know in which cases the given approach is equivalent to the discretization of stochastic continuous-time model with Brownian motion.

This thesis gives the answer with the following arguments. Given

\[
dX_t = f(t, X_t) \, dt + \sigma(t) \circ dW_t, \tag{1.17}
\]

a general FS SDE modelling a physical system, the equivalent Ito SDE can be written in the same way, once the added correction term defined by equation (1.13) does not depend on the state named \( x \). Furthermore, assuming a general time discretization \((\tau)_{\Delta}\), SDE
can be approximated by the Milstein scheme as

\[ X_{n+1}^\Delta = X_n^\Delta + f(n, X_n^\Delta)\Delta + \sigma(n)\Delta W_n, \tag{1.18} \]

where, for random number generators in computers, \( \xi_n \) are \( \mathcal{N}(\sigma(n); \Delta) \) normally distributed r.v.s with correlation \( \delta_0 \to 0 \), i.e. the traditional white noise sequence. It was also shown that the drift has no correction, i.e. Milstein and Euler-Maruyama approaches are equivalent for systems with Wiener integrals. It means that the order of convergence \( \gamma = 1.0 \) is performed by both methods and it is equivalent to that obtained by the Euler method for ODEs.

For the general case given by FS SDE (2.4) and by Ito SDE (2.2), the Milstein scheme can be rewritten as a function of the Euler-Maruyama one by defining the process \( \{U_n^\Delta\} \) given in FS sense by

\[ U_{n+1}^\Delta = X_{n+1}^\Delta + \frac{1}{2} \sigma_n(X_n^\Delta) \frac{\partial \sigma_n}{\partial x} \{((\Delta W_n)^2 + \Delta) \}. \]

Now, the terms which improve the convergence order belong to the drift and diffusion terms of the given SDE, but both still depend on the existence of the state variable \( x \) and on the diffusion function \( \sigma \). Assuming, again, that \( \sigma \) is smooth, both approximation schemes become closer.

### 1.3.2 Continuous-Time Nonlinear Filtering

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space equipped with a filtration \( \mathcal{F}_{t \geq 0} \) on which is given, for \( D, M \geq 1 \), a \( (D + M) \)-dimensional standard Brownian motion \( \{ (W_t, V_t), \mathcal{F}_t \}_{t \geq 0} \) independent on \( \pi_0 \), a \( D \)-dimensional, \( \mathcal{F}_0 \)-measurable random vector

\[ \mathbb{E}[|\pi_0|^2] < \infty. \]
The signal process \( \{X_t, \mathcal{F}_t\}_{t \geq 0} \) is the following unobserved Markov diffusion process represented by

\[
dX_t = f(t, X_t) \, dt + \sigma(t, X_t) \, dW_t, \quad X_0 \sim \pi_0, \tag{1.19}
\]

where \( f : \mathbb{R}_+ \times \mathbb{R}^D \to \mathbb{R}^D \) and \( \sigma : \mathbb{R}_+ \times \mathbb{R}^D \to \mathbb{R}^{D \times D} \) are globally Lipschitz functions. In accordance with the classical theory of SDEs in Itô sense (Liptser; Shiryayev, 1977), these hypothesis guarantee the existence and uniqueness of the solution of equation (1.19) and

\[
x(\cdot) \in L^2(\Omega; \mathcal{F}; \mathbb{P}; \mathcal{C}([0, T]; \mathbb{R}^D)),
\]

for any fixed final time \( T > 0 \).

Motivated by applications in aerospace systems (Merhav, 1996) – (Malyshev et al., 1996), the so-called observation process \( \{Y_t, \mathcal{F}_t\}_{t \geq 0} \) satisfies the Markov process represented by

\[
dY_t = h(t, X_t) \, dt + dV_t, \quad Y_0 = 0, \tag{1.20}
\]

where \( h : \mathbb{R}_+ \times \mathbb{R}^D \to \mathbb{R}^M \), and the Wiener process \( \{V_t\}_{t \geq 0} \) has the uniformly positive definite matrix \( R(t) \), a time-dependent diagonal covariance matrix. These functions are globally Lipschitz and, as a consequence of the linear growth bound of \( h \),

\[
\mathbb{E} \left[ \int_0^T |h(s, X_s)|^2 ds \right] < \infty.
\]

Let \( \mathcal{F}_0^t \) be the filtration generated by \( Y \) up to time \( t \) and \( \varphi : \mathbb{R}^D \to \mathbb{R}^M \) be a bounded continuous function. The solution \( \pi_t(\varphi) \) of the classical filtering problem, as presented by
equation (1.1), can be obtained from FS formula (Kallianpur, Striebel, 1968)

\[ \pi_t(\varphi) = \frac{\hat{E}[\varphi(X_t) \Lambda_t | \mathcal{F}_0^t]}{\hat{E}[\Lambda_t | \mathcal{F}_0^t]}, \quad \mathbb{P}\text{-a.s.,} \quad (1.21) \]

where

\[ \Lambda_t = \exp \left\{ \int_0^t h^*(s, X_s) R^{-1}(s) dY_s - \frac{1}{2} \int_0^t h^*(s, X_s) R^{-1}(s) h(s, X_s) ds \right\} \quad (1.22) \]

is the Radon-Nykodin derivative which transforms the observation process in Brownian motion (Liptser, Shiryaev, 1977) and the operator \( \hat{E} \) is the expectation with respect to this new measure.

Defining now the unnormalized representation

\[ \rho_t(\varphi) = \hat{E}[\varphi(X_t) \Lambda_t | \mathcal{F}_0^t], \quad (1.23) \]

Zakai (1969) showed that, under certain conditions (Bhatt, Kallianpur, Karandikar, 1995), \( \rho_t(\varphi) \) uniquely satisfies the following evolution equation

\[ \rho_t(\varphi) = \pi_0(\varphi) + \int_0^t \rho_s(A \varphi) ds + \int_0^t \rho_s(h^*(s, X_s) \varphi) R^{-1}(s) dY_s, \quad \text{a.s.} \quad \forall t, \quad (1.24) \]

with \( \varphi \) being the domain of the infinitesimal generator given by

\[ A \triangleq \frac{1}{2} \sum_{i,j=1}^D g_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^D f_i(s, x) \frac{\partial}{\partial x_i}, \quad (1.25) \]

where each element of the noise covariance matrix in the drift of the signal process is

\[ g_{ij} = \sigma_i(t, x) \sigma_j(t, x), \quad i, j = 1, \ldots, D. \]

Combining equations (1.21) and (1.22) and using the conditions imposed on equation (1.24), the conditional expectation (1.21) can be rewritten in terms of Feynman-Kac
formula (Del Moral, 2004)

\[
\pi_t(\varphi) = \frac{\int_{\mathcal{D}} \varphi(x') e^{U(t,x',dy_t)} p(t,x') \, dx'}{\int_{\mathcal{D}} e^{U(t,x',dy_t)} p(t,x') \, dx'}, \tag{1.26}
\]

where the potential (Feynman; Hibbs, 1965) is defined as

\[
U(t,x_t,dy_t) \triangleq \int_0^t h^*(s,x_s) R^{-1}(s) \, dy_s - \frac{1}{2} \int_0^t h^*(s,x_s) R^{-1}(s) h(s,x_s) \, ds \tag{1.27}
\]

and \(p(t,x)\) is the solution of the so-called Fokker-Planck equation

\[
\frac{\partial p}{\partial s}(s,x) + \mathcal{A}^* p(s,x) = 0, \quad p(0,x) \sim \pi_0. \tag{1.28}
\]

For real life problems, however, the solution presented in equation (1.26) may not be feasible, once the mathematical model describing the observation process (1.20) is not perfect. The robustness in this representation formula in the sense of its dependence on the process of observations is continuous was reached with the use of the so-called pathwise formulation developed in the seminal paper of Clark (1978) and after by Davis (1979).

This new formulation is quasi-deterministic in that no stochastic integration is involved. The observation sample path merely parameterizes an otherwise deterministic system of equations, where the solution of which provides the required conditional distribution. In other words, the solution of the nonlinear filtering problem can be obtained separately for each sample path of the observation process \(Y_t\).

The \(k\)th component of the signal in Itô SDE (1.19) is given by

\[
dX_t^k = J_t^k \, dt + \sum_{i=1}^D \sigma_t^{k,i} \, dW_t^k, \quad k = 1, 2, \ldots, D, \tag{1.29}
\]

with \(J_t^k \triangleq J^k(t,x)\) and \(\sigma_t^{k,i} \triangleq \sigma^k(t,x)\).

A contribution of this thesis is to rewrite the pathwise equations for filtering using a more general description of the observation disturbances. The \(m\)th component of the
observation in equation (1.20) is now given by

$$dY^m_t = h^m_t \, dt + \lambda^m_t \, dV^m_t, \quad m = 1, 2, \ldots, M,$$  \hspace{1cm} (1.30)

with $h^m_t \triangleq h^m(t,x)$ and $\lambda^m_t \triangleq \lambda^m(t)$ being the root of the correspondent diagonal elements of the covariance matrix $R(t)$.

The idea is to find an equation for the quantity $u(t, x) = \rho(t, x) \exp \{ Y_t h(t, x) \}$, instead of $\rho$ in Zakai equation (1.24). Integrating by parts the Girsanov exponential (1.22), it is possible to define a new measure $\mathbb{P}'(dz')$ in which the nonlinear filtering solution can be rewritten as follows:

$$\pi_t(\varphi) = \frac{\int_{R^D} \varphi(z') e^{[(y_s, h_s)]} e^{U(t, z', y_s)} \mathbb{P}'(dz')}{\int_{R^D} e^{[(y_s, h_s)]} e^{U(t, z', y_s)} \mathbb{P}'(dz')}, \hspace{1cm} (1.31)$$

where the potential is now defined as

$$\bar{U}(t, y_t, y_r) \triangleq \int_0^t c(r, z_r, y_r) \, ds, \hspace{1cm} (1.32)$$

with

$$c(r, z_r, y_r) = \frac{1}{2} \left( g \nabla (\bar{Y}_r, h_r) \right)^T \nabla (\bar{Y}_r, h_r) + (\bar{Y}_r, \bar{h}_r) - (\bar{Y}_r, \bar{A}_r h_r) - \frac{1}{2} (\bar{h}_r, h_r), \hspace{1cm} (1.33)$$

and

$$\bar{A}_r h^m_r = \left( \frac{\partial h^m_r}{\partial r} + \bar{A}_r h^m_r \right),$$

$$\sum_{m=1}^M \frac{1}{\lambda^m_r} h^m_r Y^m_r \triangleq \sum_{m=1}^M \bar{Y}^m_r h^m_r \triangleq (\bar{Y}_r, h_r),$$

and

$$\frac{1}{\lambda^m_r} Y^m_r \triangleq \bar{Y}^m_r,$$
where $(\cdot, \cdot)$ denotes a scalar product.

Therefore, the robust representation of the nonlinear filtering problem can be solved whenever it is possible to compute the new measure $\mathbb{P}'(dz') = p(t, z') dz'$, where $p(t, z)$ is solution of *Feynman-Kac* formula

$$
\frac{\partial p^Y}{\partial s}(s, x) = (A^Y)^* p^Y (s, x) + c^Y p^Y (s, x), \quad p(0, x) \sim \pi_0,
$$

which means an alternative signal process $Z_t$ instead of the original $X_t$.

### 1.3.3 MC Methods to Solve the Filtering Problem

The filtering theory is concerned with a recursive formula for the conditional *a-posteriori* statistical distribution for the signal based on prior observations. The obtention of recursive schemes to implement the solution of the filtering problem is of paramount importance for the application of these filters in computers. Specifically in this thesis, the numerical solution of continuous-time filtering is solved recursively by MC integration method using the discretized modelling equations.

In this text, recursive MC methods are applied to the following approaches for solving the filtering problem: 1-) the classical theory presented by Kallianpur-Striebel formula using either Zakai equation or the pathwise representation as the unnormalized conditional expectations, and 2-) the traditional Bayesian theory. In both cases, the solution of the filtering problem are numerical approximations whose results depend on the available computational effort used to simulate the weighted trajectories or *particles* of the model.

#### 1.3.3.1 MC Filters Approximating the Zakai Equation

Given the time discretization $(\tau)_\Delta$ of a bounded time interval $[0, T], \ T > 0$, the conditional expectation $\pi_t(\varphi)$ given by Kallianpur-Striebel formula (1.21) can be approximated
as follows:

\[
\pi_n^\Delta(\varphi) = \frac{\rho_n^\Delta(\varphi)}{\rho_n^\Delta(1)},
\]

(1.35)

where \(\rho_n^\Delta(\cdot)\) is the unnormalized representation.

Using Zakai equation \(\text{[ZAKAI, 1969]}\) as the unnormalized representation, the solution of the filtering problem can be rewritten in accordance with Feynman-Kac notation as follows:

\[
\pi_n^\Delta(\varphi) = \frac{\int \varphi(x_n^\Delta) \prod_{j=1}^n g_j^\Delta(x_{j-1}^\Delta, \Delta y_j) \prod_{j=1}^n \mathbb{P}_\Delta(x_{j-1}, dx_j) \mathbb{P}(dx_0)}{\int \prod_{j=1}^n g_j^\Delta(x_{j-1}^\Delta, \Delta y_j) \prod_{j=1}^n \mathbb{P}_\Delta(x_{j-1}, dx_j) \mathbb{P}(dx_0)}
\]

(1.36)

where

\[
\Lambda_{1:n}^\Delta = \prod_{j=1}^n \exp \left( \int_{x_{j-1}^\Delta}^{x_j^\Delta} (\overline{h}_j^T(x) \Delta y_j - \frac{1}{2} |\overline{h}_j(x)|^2 \Delta) \right)
\]

(1.37)

with the so-called fitness function given by

\[
g_j^\Delta(x, \Delta y) = \exp \left( \int_{x_{j-1}^\Delta}^{x_j^\Delta} (\overline{h}_j^T(x) \Delta y - \frac{1}{2} |\overline{h}_j(x)|^2 \Delta) \right),
\]

and \(\mathbb{P}_\Delta\) is the probability measure for the discretized trajectories.

The concept of random measure can be applied to the discretized solution of the filtering problem by rewriting equation (1.36) as

\[
\pi_n^\Delta(\varphi) = \int \varphi(x_n^\Delta) \mathbb{P}_\Delta^\cdot(dx_n|y_{1:n}),
\]

(1.38)

where

\[
\mathbb{P}_\Delta^\cdot(dx_n|y_{1:n}) = \sum_{i=1}^P \mu_n^{(i)} \delta_{x_n^{(i)}}(dx_{0:n})
\]
with $\mu_n^{(i)}$ being the weight corresponding to all information with respect to the particle or realization $i$, $i = 1, \ldots, P$.

Crisan and Lyons (1997) and Crisan, Gaines and Lyons (1998) constructed a measured value process whose expectation at any time is the conditional distribution of $X_t$. This is a branching particle system approximation and the particles evolve independently moving with the same law as $X$ and branch according to a mechanism that depends on the trajectory of the particle and the observation process $Y$, but is independent of the events elsewhere in the system. The result is a cloud of paths where those surviving to the current time provide an estimate for the conditional distribution of $X_t$ as follows:

$$ U_t^{P(\Delta_t)} = \frac{1}{P(\Delta_t)} \sum_{i=1}^{P(\Delta_t)} \mu_{\Delta_t; \Delta+1}^{(i)} \delta_{X(\Delta)}(\Delta+1), $$

where $P(\Delta_t)$ is the number of particles corresponding to the time instant $t = \Delta_t$.

Next corollary states about the convergence of the branching particle system, or branching particle filter (BPF), to the solution of Zakai equation.

**Corollary 1.3.2 (Crisan (2003)).** If the length of the inter-branching times is $\frac{1}{\mu}$, where $\alpha \in \left(\frac{2}{3}, 2\right)$, then

$$ \lim_{P \to \infty} P^{1-\frac{\alpha}{2}} \mathbb{E}[(U_t^P, \varphi) - \rho_t(\varphi)]^2 = c_U(t), $$

where $c_U(t)$ is a constant independent of $P$ and $\rho_t(\varphi)$ is the unnormalized conditional distribution (1.23).

The initial objective of this thesis in the study of BPFs was to establish a first contact with the use of particle methods regarding the approximation of the solutions of the nonlinear filtering problem for physical systems modelled in the continuous-time approach. The study here presented is focused on the application of these branching particles systems into simple problems aiming at studying the behavior of the methods and their main limitations.

As to be presented in Chapter 4, the implementation of BPFs as proposed by the
seminal works of Crisan and Lyons (1997) and Crisan, Gaines and Lyons (1998) has serious limitations once the branching process can either be killed or have an enormous number of realizations. This thesis, based on the preliminary work of Jacob and Yoneyama (2004), proposed the modified BPF (MBPF) in order to keep the number of particles limited: there must be established a range containing the minimum and the maximum number of particles for BPF. The decision rule is as follows: when the number of particles at time $\Delta t$ is larger than an upper limit, then the particles are not replaced for new ones; when the number of particles at time $\Delta t$ is smaller than a downer limit, then the particles are replaced for the ones with higher likelihood.

For applications in engineering, there is the need to compare the results of MC filters to the traditional approach given by the extended Kalman filtering (EKF) (Anderson, Moore, 1979). In this spirit, the study of BPF and MBPF approaches was done aiming at the development of feasible schemes for implementing the results for real-time applications. Numerical results using a one-dimensional Ornstein-Uhlenbeck process as the signal process and an observation process containing saturation showed that MC methods can obtain promising results depending on the number of particles, mainly at strong nonlinear regions. MC filters also showed promising results for controlling the estimates of the filter for unstable systems.

1.3.3.2 MC Filters Approximating the Pathwise Representation

Similarly, for the robust notation, the filtering problem is represented as follows:

$$
\pi_n^\Delta (\varphi) = \frac{\int \varphi(z^n) \exp \left[ (y_n, h_n) \right] \prod_{j=1}^{n} \tilde{g}_j^\Delta (z_j, y_j) \prod_{j=1}^{n} \mathbb{P}_Y^\Delta (z_{j-1}, d_z) \mathbb{P}_Y^\Delta (dz_0)}{\int \exp \left[ (y_n, h_n) \right] \prod_{j=1}^{n} \tilde{g}_j^\Delta (z_j, y_j) \prod_{j=1}^{n} \mathbb{P}_Y^\Delta (z_{j-1}, d_z) \mathbb{P}_Y^\Delta (dz_0)}, \quad (1.39)
$$

where

$$
\Lambda_{1:n}^\Delta = \prod_{j=1}^{n} \exp \left( c(j, X_{j-1}^\Delta, Y_j) \Delta \right) = \prod_{j=1}^{n} \tilde{g}_j^\Delta (X_{j-1}^\Delta, Y_j)
$$
with the so-called fitness function given by the deterministic term

\[ \bar{g}^\Delta (x, y) = \exp(c(j, x, y)), \]

and \( \mathbb{P}^\Delta_Y \) is the modified probability measure for the discretized trajectories depending the observation.

In other words, based on equation (1.39), the new representation of MC approximation is given by

\[ \tau_n^\Delta (\varphi) = \sum_{i=1}^{P} \tilde{\varphi}_2(z_n^{(i)}) P_n^{(i)}, \quad (1.40) \]

where

\[ P_n^{(i)} \triangleq \frac{K_n^{(i)}}{\sum_{i=1}^{P} \tilde{\varphi}_1(z_k^{(i)}) K_n^{(i)}} \]

is the normalized weight,

\[ K_n^{(i)} = \exp\left( \int_{0}^{\tau_n} c(Y(r, z_n^{(i)})) dr \right) \]

is the modulation coefficient, and

\[ \tilde{\varphi}_1(r, z) = \exp\{y_r, h(z)\} \quad \text{and} \quad \tilde{\varphi}_2(r, z) = \exp\{y_r, h(z)\} \varphi(z). \]

In papers of [Davis and Wellings (1980)] and after [Davis (1981a)] is proposed a MC method for approximating the robust representation of the nonlinear filtering solution via the pathwise notation ([Clark, 1978]–[Davis, 1981b]–[Pardoux, 1981]). An interesting point for this approach is that there are only two papers showing results about the filtering implementation: the first presented by [O’Loghlen and Wright (1982)] and the other in the doctoral thesis of [Souza (1992)]. Both papers state the high computational complexity of the method, once it requires a great number of independent realizations. However,
though nowadays these problems of computational workload are minimized, research in the filtering field has not presented contributions in the performance of this so-called pathwise filter. There is a short citation in the technical report of Del Moral (2004), but there the mathematical transcription was developed under the perspective of the use of mathematical operators and without implementation of the algorithms.

The solution suggested by Davis (1981) to simulate trajectories used ideas from Kushner (1977) which approximated PDEs using a Markov chain. It is now known that the simulation of trajectories using discretized SDE has better accuracy and less computational effort than in the Markov chain case (Kloeden; Platen, 1999). In this way, aiming at replacing the traditional method by the simulation of trajectories according to the Milstein scheme, the paper of Jacob, Hemerly and Yoneyama (2004) studied the performance of the filter using numerical analysis and compared the results with the ones obtained by EKF.

Preliminary experiments performed by Jacob and Yoneyama (2005b) showed that the pathwise MCF started diverging when the signal-to-noise ratio in the state process is not favorable, i.e. low. The time step Δ was decreased aiming at verifying the influence of discretization errors over the estimates. However, the error magnitudes did not change.

An important point to be analyzed in the robust representation is that the measure \( \mathbb{P}_Y^\Delta \) is obtained via the numerical simulation of an alternative signal process \( Z^\Delta \) whose continuous-time SDE is given by

\[
dZ_t = \mathbf{f}_Y^\Delta(t, Z_t)dt + \sigma(t, Z_t)dB_t,
\]

where \( B_t \) is a standard Brownian motion process and

\[
\mathbf{f}_Y^\Delta(s, x) = \mathbf{f}_Y(s, x) - g(s) \sum_j \frac{\partial h_j(s, x)}{\partial x_j}.
\]

This process has a drift coefficient which depends on the observation \( Y \). Assuming that
the original signal process $X$ is stable, there must be specific cases where the adjusted drift $\tilde{m}$ in the process $Z$ becomes positive, what makes the system unstable. This is an important contribution of this thesis.

### 1.3.3.3 MC Methods Applied to the Bayesian Theory

Let $x_{0:n} \triangleq \{x_0, \ldots, x_n\}$ and $y_{0:n} \triangleq \{y_0, \ldots, y_n\}$ be the samples of the discretized SDEs representing the signal and the observations up to time $n$, respectively. The aim of the filtering is to estimate recursively in time the posterior distribution $p(x_{0:n}|y_{0:n})$ and its associated features including the marginal distribution $p(x_n|y_{0:n})$, once the solution of the problem is given by

$$
\pi_{0:n}(\varphi) = \mathbb{E}[\varphi(x_{0:n})| Y_0 = y_0, \ldots, Y_n = y_n] = \frac{\int \varphi(x_{0:n}) p(x_{0:n}|y_{0:n}) dx_{0:n}}{\int p(x_{0:n}|y_{0:n}) dx_{0:n}}. \quad (1.42)
$$

Particle filters are sequential MC methods which approximate the posterior densities $p(x_{0:n}|y_{0:n})$ by swarms of points, the so-called particles, in the sample space. The particles each have an assigned weight and the posterior distribution can then be approximated by a discrete distribution which has support on each of the particles. The probability assigned to each particle is proportional to the weight.

In this thesis, the particle filters developed using Bayesian theory \cite{DOUCET2001185} were rewritten in Feynman-Kac notation under the context of the discretization of continuous-time systems.

### 1.3.4 Schemes for Improving Filtering Performance

The basic procedure for running a Monte Carlo-based filter, either using the continuous or discrete-time approaches, consists in executing three basic operations: 1-) the generation of new particles, i.e. sampling the non-observable state-space, 2-) the computation of weights associated to the particles, i.e. computing the probability masses,
and 3-) the resampling of weights aiming at renewing the information contained in a reasonable number of particles.

The particles resampling is necessary to solve the so-called weight degeneracy (LIU; CHEN, 1995). As the number of iterations increases in the filter, a large number of particles has low weight values, while a small number has very large ones; i.e. there is no particles diversification. This approach, however, reduces the accuracy of the estimate increasing the variance, what suggests that the resampling step should be applied with parsimony (LIU; CHEN; LOGVINENKO, 2001). The effective number of particles exists to check the necessity to apply the resampling. This method also difficulties the algorithm parallelization (BOLIC, 2004), once the resampling schemes need that the parallelization processing units exchange information about the sorting of the weights.

Due to this last difficulty it appears the need for an investigation of the alternatives to the resampling step trying to address the parallelization problem. A preliminary solution for this problem is given by the so-called particles restoration method which was proposed by Jacob and Yoneyama (2005a) and improved in this thesis. This open method consists in moving the particles $x_t^{(m)}$, $m = 1, 2, \ldots, \tilde{P}$, located at the tail of a given function $p(y_t|x_t^{(m)})$ towards regions where the likelihood is more significative. The moving is made by using an attractor function whose values guarantee that every sampling in $p(y_t|x_t^{(m)})$ has a correspondent application to a region where $\tilde{p}(y_t|x_t^{(m)})$, the new value, is equal or greater than the last one.

The interesting point of this new method is that the restoration uses information about the parametric structure of the model, what shows, according to the results presented by the comparison with the traditional resampling, that the particles can be better distributed. Another important point is that the methodology makes the algorithm parallelization feasible.
1.4 Outline of Thesis

Chapter 2 presents the main concepts about the continuous-time mathematical mod-
elling of physical systems. The elementary motivations with respect to the simulation
methods are shown supported by the main theoretical results in stochastic theory. The
chapter aims at presenting the mathematical notation to be used along the thesis.

The classical problem of nonlinear filtering is presented in Chapter 3. There the so-
lution of the nonlinear filtering problem, specifically Zakai equation [ZAKAI, 1969] and
the pathwise formulation (CLARK, 1978), is described according to Feynman-Kac notation
(DEL MORA, 2004). More emphasis is given in the deduction of the basic filtering equa-
tions for the pathwise filter when the observation process is disturbed by non-standard
Brownian motion.

MC methods applied to the classical nonlinear filtering equations are described in
Chapter 4. Important theorems are presented supporting the possibility to implement the
integral-based solutions of the nonlinear filtering problem using discretization schemes.
Discussions about the implementation of the branching particle filter and the interacting
particle filter are developed. The main focus is the analysis with respect to the numerical
divergence of the pathwise filter when the observation signal makes the pathwise SDE
unstable.

Chapter 5 and 6 describe MC filters based on the Bayesian notation, the so-called SIR
filters. In this approach, the filtering equations were developed in accordance with the
Feynman-Kac formula, showing the equivalence of these methods with the continuous-
time schemes. Aiming at proposing alternative methods for replacing the traditional
resampling step, Chapter 6 is focused on the preliminary description of the restoration
method, a promising scheme that can reduce the weight degeneracy and facilitate the
parallelization of the algorithm.

The conclusion and the discussion about possible future developments are presented
in Chapter 7.
2 Stochastic Modelling

2.1 Introduction

Suppose a physical system affected by certain noise-corrupted actuators and sensors. The objective of a mathematical model associated with this real system is to generate an adequate and tractable representation of the measurements behavior of the system outputs. Adequacy and tractability are subjective and are functions of the intended use of the model. The desire to develop estimators and controllers imposes a certain structure upon these models. It has already been seen that linear system models driven by white Gaussian noise are not only descriptive of many observed phenomena, but also yield a basis of useful estimator algorithms with analytical solutions (Kalman [1960]–Kalman; Bucy [1961]). Controller design based on such a model is also tractable and useful in practical applications (Goodwin; Sin [1984]).

Physical systems can be represented by mathematical models using 1-) principles of Physics, the so-called theoretical or analytical models, and 2-) data from direct observation of the process, the empirical or heuristical models (Faleiros; Yoneyama 2002).

A well acceptable procedure to determine mathematical models is usually based on the following steps (Garcia 1997):

1. System specification based on characteristics of the physical model
   - simplifications are assumed and input and output variables are defined.

2. Description of ODE-based mathematical model
• selection of variables which describe the instantaneous states of the system,
• identification of possible relations among variables, e.g. balance and compatibility,
• identification of constitutive relations for sets of variables,
• combination of the obtained relations.

3. Investigation of the system dynamic behavior by using solutions of ODEs via analytical or numerical methods.

The engineer must not forget that the model being analyzed is only an approximate mathematical description of the system, not the physical system itself. Conclusions based on equations that required a variety of assumptions and simplifications in their development may or may not apply to the actual system. Unfortunately, the more faithful a model is in describing the actual system, the more difficult is to obtain general results (CLOSE; FREDERICK, 1995).

The modelling of plants often uses physical laws traditionally developed for continuous-time systems, as ODEs-based Newton’s and Kirchhoff’s theories for describing mechanical and electrical systems, respectively. Differential equations governing real processes always contain some elements, e.g. coefficients and inhomogeneous part, which characterize physical features of the phenomenon/environment and are experimentally determined. Due to errors in the measurements and inherent randomness of phenomena, these elements cannot most often be expressed by one single defined function $f$, but they have to be characterized by a family of functions $f_\omega$ depending on a certain parameter $\omega$. Usually, however, the engineer is not able to foresee which of these functions $f$ have to be replaced by a random function $f_\omega$, where now the parameter $\omega$ is interpreted as an element of the space of elementary events $\Omega$ on which a probability is defined.

In fact, the lack of knowledge about the exact behavior of the system along with possible natural statistical fluctuations, e.g. shot noise in electronics, are modelled by adding to the respective state a random force whose finite-dimensional distributions are
supposed to be known, the so-called external noise (Van Kampen, 1981). Stochastic
problems in engineering are of this type. Such cases are described by SDEs and present
no physical but merely mathematical problems.

The inclusion of r.v.s in ODEs describing the behavior of a plant leads to two distinct
classes of equations whose solution processes have differentiable and nondifferentiable
sample paths, respectively. The first and simpler class arises when an ODE has: 1-)
random coefficients, 2-) a random initial value, or 3-) is forced by a fairly regular stochastic
process, e.g. processes with differentiable sample paths, or when some combination of these
holds (Sussmann 1978). The sample paths of the solution processes are then at least
differentiable functions. For this situation, the general equation is given by

$$\dot{X} = \frac{dX}{dt} = f(t, X_t) + \sigma(t, X_t) \xi_t, \quad (2.1)$$

where $f : [0, T) \times \mathbb{R}^D \rightarrow \mathbb{R}^D$ and $\sigma : [0, T) \times \mathbb{R}^D \rightarrow \mathbb{R}^D$ and the forcing process $\{\xi_t\}_{t \geq 0}$
has differentiable sample paths or realizations. Given a realization of the initial value of
the state process, which is a r.v. of known distribution $\pi_0$, the solution of RDE (2.1) is
obtained by ordinary calculus integrating each sample path independently.

A virtual problem in physics and engineering is to model random phenomena, in par-
ticular, in the theory of dynamical systems, stochastic inputs and stochastic observation
errors. These, for reasons that include mathematical convenience, the forcing term is
a stochastic process with fairly irregular sample paths, e.g. $\alpha$-Hölder continuous with
$\alpha \in (0, 1)$ such as the Brownian motion. Under this stipulation the problem becomes
that of defining a mathematical object, that might be regarded as reasonable model of
the so-called white noise, and whereby a rigorous and applicable theory of SDEs might
be constructed (d’Alejandro 1980).

In the second class, RDE (2.1) is then rewritten formally as stochastic differential which
is interpreted as an integral equation composed by Ito (Ito 1944) or Fisk-Stratonovich
(FS) (Fisk, 1963)-(Stratonovich, 1964) stochastic integrals. This equation is a SDE
and in general its solution inherits the nondifferentiability of sample paths from the Brownian motion in the stochastic integrals \cite{KLOEDEN2019}. The task of giving a precise mathematical meaning to SDEs is thus transformed into the task to define rigorously what these two integrals really mean.

Ito and FS integrals lead to a consistent calculus. The question is, how to interpret a SDE with Gaussian white noise that has been obtained via some limiting procedure, or to rephrase the question, what are the coefficients of the diffusion process that model most adequately the system being described? Physical arguments have to be invoked to determine the appropriate drift and diffusion coefficient of the process. Formulated in this way, it becomes clear that Ito versus FS approaches are void \cite{VANKAMPEN1981}, once a diffusion process can be described either way \cite{JAZWINSKI1970}.

This chapter clarifies some issues in the mathematical modelling of continuous-time systems with uncertainties of stochastic type. Here are presented arguments stating that once the states of a plant evolve continuously in time, the modelling of uncertainties is more natural and realistic when the associated processes are also continuous in time. However, from a mathematically rigorous viewpoint, modelling the stochastic uncertainties in continuous-time may lead to two different interpretations: Ito and FS ones. For a narrow class of systems, e.g. models with additive noise independent of the state, both approaches are equivalent and the traditional discretized approach is obtained with the definition of white-noise. But, for a more general class, the system equation has a difference on its drift, without any physical motivation \cite{HORSTHEMKE1984}, which must be considered during the computations.

Next sections concern about clarifying important issues in the mathematical modelling of continuous-time systems with Brownian motion. The idea is to present the main results in stochastic theory which demonstrate the effectiveness of FS approach, instead of the Ito one, in modelling physical systems and in simulating their respective equations using computers. In both cases, it is shown that stochastic integrals based on very small correlation uncertainties, e.g. those ones obtained in physical processes and
computer-generated random numbers, converge to FS integral in mean-square. Taylor-
type approximation schemes, as the Euler-Maruyama and the Milstein ones, are presented
to illustrate discretization methods which allow the simulation of continuous-time math-
ematical models in computers. Finally, examples discuss the mathematical conditions in
which the method of including random fluctuations in ODEs, an easy but conceptually
wrong method for modelling plants with uncertainties, is equivalent to the discretization
of stochastic continuous-time model with Brownian motion.

2.2 Brownian Motion and White Noise

Noise as an idea came upon the scene with a power and swiftness that transformed all
of science and the men’s views of the nature of matter (COHEN 2005). In the eighteenth
century the famous experiment of Robert Brown about the trajectories performed by
molecules described qualitatively the existence of the so-called Brownian motion. The
first mathematical transcription of this phenomenon was proposed by Albert Einstein and
Paul Langevin in the beginning of the last century, but just in the 1930’s the mathematics
was finally succeeded with Paul Levy and Norbert Wiener in producing a precise formal
theory (WAX 1954).

A well-accepted mathematical transcription of the Brownian motion phenomenon as a
stochastic process was suggested by Norbert Wiener. This process, the so-called Wiener
process, is basic to the construction of mathematical models of random phenomena in an
amazing range of applications.

Definition 2.2.1. An R-valued stochastic process \( \{W_t\}_{t \geq 0} \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\)
is a scalar standard Wiener process when

- (i) \( \{W_t\}_{t \geq 0} \) is null at the origin;

- (ii) the sample-paths \( t \mapsto W_t(\omega) : [0, \infty) \rightarrow \mathbb{R} \) are continuous for each \( \omega \in \Omega \);
(iii) for each $s, t \in [0, \infty)$ with $s < t$, the increments $W_t - W_s$ are Gaussian distributed with mean zero and variance $t - s$;

(iv) $\{W_t\}_{t \geq 0}$ has independent increments.

The last definition is amply motivated on physical grounds and completely satisfactory for the requirements of elementary and classical probability theory (PAPOUlis [1965]). The Wiener process has played a major role in the mathematical formulation of many problems in filtering, communication and stochastic control theory. This is mainly due to the availability of powerful Ito stochastic calculus which enables explicit answers to be given to questions which, if not formulated in terms of Brownian motion, would be quite intractable (DAVIS [1978]).

From the point of view of stochastic calculus however, Definition 2.2.1 has one significant disadvantage, namely it includes no reference to a filtration in $(\Omega, \mathcal{F}, \mathbb{P})$. In this way, an equivalent definition of a Wiener process in terms of an underlying filtration must be formulated.

**Remark 2.2.2.** For every stochastic process $X_t$, the information that has been revealed up to time $t$ corresponds to the intuitive notion of filtration denoted by $\mathcal{F}_t^X = \sigma(X_s; 0 \leq s \leq t)$. The resulting family $\{\mathcal{F}_t^X; 0 \leq t < \infty\}$ is increasing: $\mathcal{F}_t^X \subseteq \mathcal{F}_s^X$ for $t < s$.

**Definition 2.2.3.** An $\mathbb{R}$-valued stochastic process $\{W_t\}_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a scalar standard Wiener process when (i), (ii) and (iii) of Definition 2.2.1 hold, and (iv) for each $s, t \in [0, \infty)$ with $s < t$ the increment $W_t - W_s$ being independent of $\mathcal{F}_s^W \triangleq \sigma\{W_u; u \in [0, s]\}$.

**Remark 2.2.4.** If $\{W_t\}_{t \geq 0}$ is a scalar standard Wiener process, then $\mathbb{E}[W_t] = 0$, $\forall t \geq 0$, and hence the covariance is

$$\text{cov}(W_s, W_t) \triangleq \mathbb{E}[W_s W_t] = s \wedge t, \ \forall s, t \geq 0,$$

where $\mathbb{E}$ is the so-called expectation.
Remark 2.2.5. Clearly any scalar stochastic process \( \{W_t\}_{t \geq 0} \) is adapted to a filtration \( \{\mathcal{F}_t, t \geq 0\} \) if and only if \( \mathcal{F}_t^W \subset \mathcal{F}_t \) for each \( t \geq 0 \). It follows that if \( \{W_t\}_{t \geq 0} \) is a scalar Wiener process in the sense of Definition 2.2.3, then \( W_t - W_s \) is independent of \( \mathcal{F}_s \), hence independent of \( \mathcal{F}_s^W \subset \mathcal{F}_s \), for each \( s, t \geq 0 \) with \( s < t \), so that the process has all the properties given in Remark 2.2.4.

Remark 2.2.6. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. A Wiener process \( W = \{W_t\}_{t \geq 0} \) containing \( \{\mathcal{F}_t, t \geq 0\} \) as an increasing family of sub \( \sigma \)-algebras of \( \mathcal{F} \) such that \( W_t \) is \( \mathcal{F}_t \)-measurable has the following characteristics

\[
\begin{align*}
(i) \hspace{.5cm} & \mathbb{E}[W_t | \mathcal{F}_0] = 0; \\
(ii) \hspace{.5cm} & \text{Martingale property:} \\
& \mathbb{E}[W_t - W_s | \mathcal{F}_s] = 0 \text{ w.p.1, } \forall s \in [0,t].
\end{align*}
\]

Definition 2.2.7. Physically the term band-limited white noise describes random fluctuations \( \{\xi_t\}_{t \geq 0} \) where:

\[(i) \hspace{.5cm} \text{at each time instant } t, \xi_t \text{ is approximately Gaussian distributed;} \]

\[(ii) \hspace{.5cm} \xi_t \text{ and } \xi_s \text{ are effectively uncorrelated for } |t - s| > \delta_0 \text{ with the so-called noise bandwidth } \delta_0 \text{ much less than the highest physical system time-constant.} \]

Remark 2.2.8. In the absence of precise information about \( \delta_0 \), which is commonly the case in practice, it has been customary in the engineering literature to translate the concept of “small” \( \delta_0 \) to \( \delta_0 \to 0 \), that is infinite bandwidth, and then refer to this process as white noise.

The process \( \{\xi_t\}_{t \geq 0} \) is a fictitious process which can be put on a rigorous basis using the Brownian motion concept when \( \delta_0 \to 0 \). Defining

\[ W_t \triangleq \int_0^t \xi_s \, ds, \]
then
\[
\text{cov}(W_t, W_s) = \int_0^t \int_0^s \mathbb{E}[\xi_u \xi_v] \, dudv = \int_0^t \int_0^s \delta(u - v) \, dudv = t \wedge s,
\]

which means that the process is a Brownian motion.

**Definition 2.2.9.** Brownian motion is the indefinite integral of the white noise.

Although Gaussian white noise is so very irregular, it is extremely useful to model rapidly fluctuating phenomena. Not surprisingly, in view of its properties, e.g. no continuous sample paths and infinite total power, true white noise does not occur in nature (Davis [1977]). However, as can be seen by studying their spectra, thermal noise in electrical resistance, the force acting on a Brownian particle, or climate fluctuations, disregarding the periodicities of astronomical origin, are white to a very good approximation. These examples support the usefulness of the white-noise idealization in applications to natural systems. In any case, frequencies that are very much higher than the characteristic frequency of the system should be of no importance for the macroscopic properties of the system which, due to its inertia, acts so to speak as a low-pass filter. Noises with a very large effective bandwidth should be indistinguishable from white noise for practical purposes. Assuming that \( \tau_{\text{cor}} \) is the typical time constant of the physical system, if \( \tau_{\text{cor}} << \delta_0 \) then it will be permissible to pass to the white-noise limit [Horsthemke; Leefver [1984]].

### 2.3 Stochastic Differential Equations

Replacing the forcing process \( \{\xi_t\}_{t \geq 0} \) in RDE (2.1) by the standard Wiener process \( \{W_t\}_{t \geq 0} \), the general equation, now a SDE, is given in the differential formulation by

\[
dX_t = \bar{f}(t, X_t) \, dt + \sigma(t, X_t) \, dW_t,
\]
or in the equivalent integral formulation by

\[ X_t = X_{t_0} + \int_{t_0}^{t} \bar{f}(s, X_s) \, ds + \int_{t_0}^{t} \sigma(s, X_s) \, dW_s, \]  

(2.3)

where the first integral, with \( \bar{f} \) generally different from \( f \) in (2.1), is an ordinary Riemann or Lebesgue integral for each sample function but the second one, the *stochastic integral*, requires some care in relation to its definition.

The stochastic integral presented in equation (2.3) is interpreted in the mean square sense according to distinct definitions given in the seminal papers of Ito (1944), the so-called Ito integral, and [Fisk (1963)] and [Stratonovich (1964)] which independently proposed the Fisk-Stratonovich integral. Particularly, for mathematical notation, the referred stochastic integral is interpreted in Ito sense and equation (2.2) is called Ito SDE.

In contrast, when the stochastic integral in equation (2.3) is interpreted in FS sense, the so-called FS SDE must be defined and it requires a new mathematical notation given by

\[ dX_t = f(t, X_t) \, dt + \sigma(t, X_t) \circ dW_t, \]  

(2.4)

or equivalently by

\[ X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s) \, ds + \int_{t_0}^{t} \sigma(s, X_s) \circ dW_s, \]  

(2.5)

where now \( f \) is the same as the one presented in RDE (2.1).

### 2.3.1 Time Discretizations and Approximations

The study of the adequate time discretization which approximates solutions of integrals has been widely used in the development of important results in the calculus theory. The stochastic calculus, in the same way, is also based on use the use of similar approximations.
Chapter 2. Stochastic Modelling

Definition 2.3.1. The equidistant time discretization \((\tau)_\Delta = \{\tau_n : n = 0, 1, \ldots\}\) of a bounded time interval \([t_0, T]\) has discretization times

\[
\tau_n = t_0 + n\Delta
\]  \hspace{1cm} (2.6)

for a constant step size \(\Delta \in (0, \Delta_0)\) for some finite \(\Delta_0 > 0\).

Let \(n_t\) be the largest integer \(n\) for which \(\tau_n\) does not exceed a fixed \(T\), i.e.

\[
n_t = \max\{n = 0, 1, \ldots : \tau_n \leq T\}.
\]  \hspace{1cm} (2.7)

Then, the time discretization \((\tau)_\Delta\) for a given maximum step size \(\Delta \in (0, \Delta_0)\) can be seen as a sequence of time instants satisfying

\[
0 \leq \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \infty,
\]

where

\[
\sup_n (\tau_{n+1} - \tau_n) \leq \Delta \quad \text{and} \quad n_t \leq \infty, \ w.p.1,
\]

for all \(t \in \mathbb{R}^+\) and with \(\tau_{n+1}\) being \(\mathcal{F}_{\tau_n}\)-measurable for each \(n\).

Definition 2.3.2 (Kloeden and Platen (1999)). A right continuous process \(\{X_t\}_{t \geq 0}\) with left hand limits is a time discrete approximation with maximum step size \(\Delta \in (0, \Delta_0)\) if: 1-) it is based on a time discretization \((\tau)_\Delta\) such that \(X_{\tau_n}\) is \(\mathcal{F}_{\tau_n}\)-measurable, and 2-) \(X_{\tau_{n+1}}\) can be expressed as a function of \(X_{\tau_0}, \ldots, X_{\tau_n}, \tau_0, \ldots, \tau_n, \tau_{n+1}\) and a finite number \(k\) of \(\mathcal{F}_{\tau_{n+1}}\)-measurable random variables \(Z_{\tau_{n+1},j}\) for \(j = 1, \ldots, k\) for each given \(n\).

Remark 2.3.3. This allows the recursive computation of the approximated values at the given \((\tau)_\Delta\). \(X_{\tau_n}\) is a r.v. restricted to be \(\mathcal{F}_{\tau_n}\)-measurable since the computation of its values should not involve more information than is available at time \(\tau_n\). The value of \(X_{\tau_{n+1}}\) may then depend on the values of \(X\) at earlier discretization times, on the step size and on a finite number of r.v.s generating the noise mainly within the current time step.
Remark 2.3.4. As an example, a Markov chain on a discrete space can be written as $X_{\tau_{n+1}} = X_{\tau_n} + Z_{\tau_{n+1,1}}$, where $Z_{\tau_{n+1,1}}$ is a r.v. characterized by the transition probabilities of the chain.

2.3.2 Ito integral

The main purpose of Ito’s work on SDE was to construct Markov processes such that their transition probabilities would satisfy given Kolmogorov equations [Wong; Zakai, 1965]. To achieve this purpose, Ito (1944) generalized the Wiener integral, i.e. integrals of deterministic square-integrable functions with respect to the Brownian motion, to include the case where the integrand is also a r.v. Here, some basic definitions and properties of the Ito integral will be shown, but its existence is not proved (see, for example, Doob (1953, Ch. 9) and d’Alessandro (1980)).

Definition 2.3.5 (Kloeden and Platen (1999)). Assuming $0 < T < \infty$, the class $L^2_T$ of functions $\sigma : [0, T] \times \Omega \to \mathbb{R}$ is defined satisfying

(i) $\sigma$ is jointly $\mathcal{L} \times \mathcal{F}$-measurable;

(ii) $\int_0^T \mathbb{E}[\sigma(t, \cdot)^2] \, dt < \infty$;

(iii) $\mathbb{E}[\sigma(t, \cdot)^2] < \infty$ for each $0 < T < \infty$;

(iv) $\sigma(t, \cdot)$ is $\mathcal{F}_t$-measurable for each $0 < T < \infty$.

Remark 2.3.6. Two functions in $L^2_T$ are considered identical if they are equal for all $(t, \omega)$ except possibly on a subset of $\mu_L \times \mathbb{P}$-measure zero. Then with the norm

$$||\sigma||_{2,T} \triangleq \sqrt{\int_0^T \mathbb{E}[\sigma(t, \cdot)^2] \, dt},$$

$L^2_T$ is a complete normed linear space (Banach space) provided just functions which differ on sets of measure zero (Hilbert space).
For any partition \((\tau)_\Delta\) given by \(0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_{n+1} = T\) and any mean-square integrable \(\mathcal{F}_{t_j}\)-measurable random variables \(\sigma_j, j = 1, 2, \ldots, n\), it is possible to define a step function \(\sigma \in \mathcal{L}^2_T\) by \(\sigma(t, \omega) = \sigma_j(\omega)\), w.p.1, for \(t_j \leq t < t_{j+1}\) and \(j = 1, 2, \ldots, n\). By denoting \(\mathcal{S}_T^2\) the subset of all step functions in \(\mathcal{L}^2_T\), it is possible to approximate any function in \(\mathcal{L}^2_T\) by step functions \(\mathcal{S}_T^2\) to any desired degree of accuracy given by norm \((2.8)\).

**Definition 2.3.7.** Let \(\sigma_j \in \mathcal{S}_T^2, j = 1, \ldots, n\), be a set of r.v.s corresponding to a partition \((\tau)_\Delta\) given by \(0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{n+1}^{(n)} = T\) with evaluation points \(s_j^{(n)} \in [t_j^{(n)}, t_{j+1}^{(n)}]\) for which

\[
\Delta_j^{(n)} = \max_{0 \leq j \leq n} (t_j^{(n)} - t_j^{(n)}) \to 0 \text{ as } n \to \infty.
\]

The generalized sum of Riemann-Stieltjes for \(\sigma_j\) over the interval \([0, T]\) is given as follows:

\[
S_n(\sigma)(\omega) = \sum_{j=0}^{n} \sigma_j^{(n)}(\omega) \Delta W_j(\omega), \text{ w.p.1,} \tag{2.9}
\]

where

\[
\Delta W_j(\omega) = W_{t_{j+1}}(\omega) - W_{t_j}(\omega), \text{ } W_0(\omega) = 0. \tag{2.10}
\]

**Remark 2.3.8.** From Definition \((2.2.3)\) the increments provided by equation \((2.10)\) in the partition \((\tau)_\Delta\) are independent distributed r.v.s obeying \(\Delta W_j \sim \mathcal{N}(0; \Delta), j = 0, \ldots, n\).

The evaluation points presented in Definition \((2.3.7)\) can be systematically rewritten as a convex function given by

\[
s_j^{(n)} = (1 - \lambda)t_j^{(n)} + \lambda t_{j+1}^{(n)}, \text{ } j = 0, \ldots, n - 1, \tag{2.11}
\]

for the same fixed \(0 \leq \lambda \leq 1\).
Definition 2.3.9 (Ito Integral). A stochastic process \( \{ \sigma_t \}_{t \in [0,T]} \) is said to be Ito integrable with respect to the standard Wiener process \( \{ W_t \} \) if the mean-square limit of the sums in equation (2.9) exists at the evaluation point \( \lambda = 0 \) in equation (2.11), that is

\[
I(\sigma) \triangleq \lim_{\Delta \to 0} \frac{1}{n} \sum_{j=0}^{n} \sigma_j \Delta W_j.
\] (2.12)

Moreover, this limit defines Ito integral \( \int_0^T \sigma_t \, dW_t \).

Remark 2.3.10. The variable \( X \) is the mean square limit of the process \( \{ X_n \} \), or \( X \) is the limit in the mean of \( \{ X_n \} \), when

\[
(X = \text{l.i.m.} \, X_n) \iff \left( \lim_{n \to \infty} \mathbb{E} \{ |X - X_n|^2 \} = 0 \right).
\]

Remark 2.3.11. Ito process itself is a time discrete approximation.

Lemma 2.3.12. For any \( \sigma, \sigma^{(1)}, \sigma^{(2)} \in S_T^2 \) and \( \alpha, \beta \in \mathbb{R} \), the Ito stochastic integral given by equation (2.12) satisfies:

(i) \( I(\sigma) \) is \( \mathcal{F}_T \)-measurable,

(ii) \( \mathbb{E}[I(\sigma)] = 0 \),

(iii) \( \mathbb{E}[I(\sigma)^2] = \int_0^T \mathbb{E}[\sigma(t, \cdot)^2] \, dt \),

(iv) \( I(\alpha \sigma^{(1)} + \beta \sigma^{(2)}) = \alpha I(\sigma^{(1)}) + \beta I(\sigma^{(2)}) \), \( \text{w.p.1} \).

Here the \( \sigma \)-algebra \( \mathcal{F}_t \) is interpreted as a collection of events that are detectable prior to or at time \( t \), so that the \( \mathcal{F}_t \)-measurability of \( X_t \) for a stochastic process \( \{ X_t \}_{t \geq 0} \) indicates its nonanticipativeness with respect to the Wiener process \( W = \{ W_t \} \).

Ito integral has very interesting properties, e.g. it is linear, martingale, etc., but it does not satisfy the change of variables formula of the ordinary calculus as shown in the theorem below.
Theorem 2.3.13 (Ito formula). Let $U : [0, T] \times \mathbb{R}^D \to \mathbb{R}$ have continuous partial derivatives $\frac{\partial U}{\partial t}$, $\frac{\partial U}{\partial x}$, and $\frac{\partial^2 U}{\partial x^2}$ for $k = 1, 2, \ldots, D$. Let $\{Y_t\}_{t \in [0, T]}$ be a scalar process defined by $Y_t = U(t, X_t)$, where $X_t$ satisfies Ito SDE (2.2) for $0 \leq t \leq T$. Then

$$dY_t = \left\{ \frac{\partial U}{\partial t} + \sum_{k=1}^{D} \int_t^k \frac{\partial U}{\partial x_k} dt + \frac{1}{2} \sum_{k=1}^{D} (\sigma_k)^2 \frac{\partial^2 U}{\partial x_k^2} \right\} dt + \sum_{k=1}^{D} \sigma_k \frac{\partial U}{\partial x_k} dW_t, \text{ w.p.1.} \quad (2.13)$$

Remark 2.3.14. In the ordinary calculus, the chain rule is the same as presented by equation (2.13) without the correction term in the drift $\bar{f}$ of the diffusion process $\{Y_t\}$.

Another important property is that for a variable subinterval $[t_0, t] \subseteq [0, T]$ it is defined the stochastic process $\{Z_t\}_{t \in [t_0, T]}$ given by

$$Z_t(\omega) = \int_{t_0}^{t} f(s, \omega) dW_s(\omega), \text{ w.p.1.} \quad (2.14)$$

Theorem 2.3.15. For $t_0 \leq s \leq t \leq T$,

$$\mathbb{E}[Z_t - Z_s|\mathcal{F}_s] = 0, \text{ w.p.1,} \quad (2.15)$$

which means that $Z$ is a martingale, that is, $\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$.

Theorem 2.3.16. A separable, jointly measurable version of $Z_t$ defined by equation (2.14) for $t \in [t_0, T]$ has, almost surely, continuous sample paths.

The martingale property $\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$ and its useful technical consequences is one of the most advantageous features of Ito stochastic integral. The price to be paid is that the stochastic differentials cannot be transformed according to the chain rule of classical calculus. Instead an additional term appears and the resulting expression is called Ito formula. Roughly speaking, the difference is due to the fact that the stochastic differential $(dW_t)^2$ is equal to $dt$ in the mean-square sense.
2.3.2.1 Uniqueness of Ito SDE Solution

The solution of the process $X = \{X_t\}_{t \in [t_0, T]}$ given by Ito SDE (2.2) must have properties which ensure that both types of integrals are meaningful.

**Definition 2.3.17.** The basic set of assumptions which guarantees the description of the solution of Ito SDE is given in what follows. The initial instant $0 \leq t \leq T$ is arbitrary, but fixed, and the coefficients functions $\bar{f}, \sigma : [t_0, T] \times \mathbb{R}^D \rightarrow \mathbb{R}^D$ are given.

(i) (measurability): $\bar{f}(t, x)$ and $\sigma(t, x)$ are jointly $L^2$-measurable in $(t, x) \in [t_0, T] \times \mathbb{R}^D$;

(ii) (Lipschitz condition): There exists a constant $K > 0$ such that

$$|\bar{f}(t, x) - \bar{f}(t, y)| \leq K|x - y| \quad \text{and} \quad |\sigma(t, x) - \sigma(t, y)| \leq K|x - y|$$

for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}^D$;

(iii) (linear growth bound): There exists a constant $K > 0$ such that

$$|\bar{f}(t, x)|^2 \leq K^2(1 + |x|^2) \quad \text{and} \quad |\sigma(t, x)|^2 \leq K^2(1 + |x|^2)$$

for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}^D$;

(iv) (initial value): $X_{t_0}$ is $\mathcal{F}_{t_0}$-measurable with $\mathbb{E}[|X_{t_0}|^2] < \infty$.

This follows when the coefficient functions $\bar{f}$ and $\sigma$ are sufficiently regular and when the process $X_t$ is regular and nonanticipative with respect to the Wiener process $\{W_t\}_{t \in [t_0, T]}$, i.e $\mathcal{F}^*$-adapted where $\mathcal{F}^* = \{\mathcal{F}_t, t_0 \leq t \leq T\}$ is the family of $\sigma$-algebras associated with the Wiener process. In addition, the integrals in (2.3) should exist, at least w.p.1, for each $t \in [t_0, T]$. Then the process $X$ is called a solution of Ito SDE (2.2) on $t \in [t_0, T]$. For fixed coefficients $\bar{f}$ and $\sigma$, any solution $X$ will depend on the particular value $X_{t_0}$ and the Wiener process $W$ under consideration. If there is a solution for each given Wiener process it is said that SDE has a strong solution. Such solution can be thought as a
functional of the initial value $X_{t_0}$ and of the values $W_t$ of the Wiener process over the subinterval $t_0 \leq t \leq T$. For a specified initial value $X_{t_0}$, the \textit{uniqueness} of solutions of Ito SDE (2.2) refers to the equivalence in probability of the solution processes satisfying the equation itself. If there is a solution, there will be a separable version which has, almost surely, continuous sample paths.

\textbf{Definition 2.3.18.} If any two solutions $X$ and $\tilde{X}$ have, almost surely, the same sample paths for all $t \in [t_0, T]$, that is, if

$$\mathbb{P} \left( \sup_{0 \leq t \leq T} |X_t - \tilde{X}_t| > 0 \right) = 0,$$

then the solutions of Ito SDE (2.2) are \textbf{pathwise unique}.

\textbf{Theorem 2.3.19 (Kloeden and Platen (1999)).} Under assumptions of Definition 2.3.17, Ito SDE (2.2) has a pathwise unique strong solution $X_t$ on $[t_0, T]$ with

$$\sup_{0 \leq t \leq T} \mathbb{E} \left( |X_t|^2 \right) < \infty.$$

\textbf{Theorem 2.3.20.} The unique solution is a Markov process on the interval $[t_0, T]$ whose initial probability distribution at $t = t_0$ is the distribution of $X_t$ and whose transition probability is given by

$$\mathbb{P}(s, y; t, A) = \mathbb{P}\{X_t \in A | X_s = y\}.$$

\textbf{Theorem 2.3.21.} If, additionally, the functions $\bar{f}$ and $\sigma$ are continuous with respect to $(t, x)$, the solution $X$ is a \textbf{diffusion Markov process} on $[t_0, T]$ with drift $\bar{f}(t, x)$ and diffusion matrix $G(t, x) \triangleq \sigma(t, x)\sigma^\ast(t, x)$.

\textbf{Remark 2.3.22.} For Theorem 2.3.21 in particular, if the coefficients of Ito SDE do not depend on $t$, then the solution $X$ is a homogeneous diffusion process.
2.3.3 Fisk-Stratonovich integral

According to Clark (1973), one of Stratonovich’s original aims in formulating his integral was to obtain a natural idealization of ODEs of the type given by RDE (2.11), where $\xi_t$ is a random disturbance that behaves like “white noise”, i.e. its integral $\int_0^t \xi_s \, ds$ behaves like Brownian motion.

Analogously to the definition of Ito integral, but now assuming the symmetric case for the evaluation points in the sum of Riemann-Stieltjes (2.9), this resultant integral was constructed by Fisk (1963) and Stratonovich (1964) in different contexts.

**Definition 2.3.23 (FS integral).** A stochastic process $\{\sigma_t\}_{t \in [0,T]}$ is said to be FS integrable with respect to the standard Wiener process if the mean-square limit of the sums in equation (2.9) exists at the evaluation point $\lambda = 1/2$ in equation (2.11), that is

$$FS(\sigma) \triangleq \lim_{\Delta \to 0} \sum_{j=0}^n \sigma_{\alpha_j} \Delta W_j.$$  \hspace{1cm} (2.16)

where $\alpha_j = t_j^{(n)} + \frac{1}{2} \Delta$. Moreover, this limit defines FS integral $\int_0^T \sigma_t \, dW_t$.

**Remark 2.3.24.** It is out of the scope of this thesis to discuss whether the limits of Definitions (2.3.9) and (2.3.23) are or are not valid. Main results about the general stochastic theory are reported in Lipster and Shiryaev (1977) and Krishnan (1984).

The major advantage of FS integral is that it obeys the usual transformation rules of calculus. However, it does not satisfy the powerful martingale properties of their Ito counterparts. Next proposition gives the relation between Ito and FS integrals. Without loss of generality, to simplify the notation, only scalar processes are now considered.

**Proposition 2.3.25.** Let $\{W_t\}_{t \geq 0}$ be a Wiener process with an associated filtration given by $\{\mathcal{F}_t, t \geq 0\}$. Let $g : [0, T) \times \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ and $\{X_t\}_{t \geq 0}$ be a diffusion process that is solution of Ito SDE (2.2). Then, Ito and FS integrals are related as

$$\int_0^T g_s(X_s) \, dW_s = \int_0^T g_s(X_s) \, dW_s - \frac{1}{2} \int_0^T \sigma_s(X_s) \frac{\partial g_s}{\partial x}(X_s) \, ds.$$ \hspace{1cm} (2.17)
Remark 2.3.26. As an immediate consequence of Proposition 2.3.25, let \( f : [0, T) \times \mathbb{R} \to \mathbb{R} \) and \( \sigma : [0, T) \times \mathbb{R} \to \mathbb{R} \) be functions of class \( C^1 \) and \( \{X_t\}_{t \geq 0} \) be the solution of the scalar Ito SDE (2.2). Then the same process is also the solution of FS SDE (2.4) with

\[
\bar{f}(t, x) \overset{\Delta}{=} f(t, x) - \frac{1}{2} \sigma(t, x) \frac{\partial \sigma}{\partial x}(t, x).
\]  

(2.18)

2.3.4 Ito versus FS SDEs

Effectively, in Ito SDE (2.2) the Brownian motion does not really exist as a physical random process. These difficulties are man-made and disappear when one takes into account that a random force in Physics is never really white noise, but at best has a very short auto-correlation time (VAN KAMPEN [1981]), i.e. the band-limited white noise. The main question arises: In what sense, if any, does the solution of a SDE approximate a physical process?

A physical system can be modelled according to the Langevin-type equation described by RDE (2.1), except that the forcing process \( \xi_t \) now is a Gaussian white noise. Since white noise is not an ordinary stochastic process the above equality is something as a “pre-equation” which should be properly interpreted in accordance with the stochastic calculus. The questions which arise are: 1-) how should RDE (2.1) be interpreted? 2-) how is the possible interpretation of the given equation with respect to Ito SDE (2.2)?

What one really means by “pre-equation” in practice is most likely an equation driven by a stationary Gaussian process whose spectral density is flat over a sufficiently wide band of frequencies (SOBCZYK [1991]). It means that RDE (2.1) can be interpreted for sample functions, provided that the spectral density of \( \xi_t \) tends to zero rapidly enough to assure regularity of the sample functions. However, such an interpretation does not lead to an effective probabilistic theory.

An interpretation mathematically attractive can be given assuming that a sequence of Gaussian processes \( \{\xi^n_t\} \), \( n = 1, 2, \ldots \), converges in some sense to a white Gaussian...
noise and the processes $\xi^n_t$ have regular sample functions for each $n$. Then for each $n$ the “pre-equation” can be solved, given that the sample function solution exists and it is unique. In this way, a sequence of processes $\{X^n_t\}$ is obtained. Suppose now that a sequence of regular processes $\{\xi^n\}$ converges in some sense to Gaussian white noise as $n \to \infty$, and the corresponding sequence $\{X^n_t\}$ converges to a process $X_t$. Therefore, it is natural to expect that $X_t$ is governed by the “pre-equation” originally proposed by RDE (2.1).

To obtain an answer to the second question it is necessary to perform precisely the limiting procedures indicated above by using discrete approximations and then to obtain an equation for the limiting process $X_t$. Such an analysis was provided in Wong and Zakai (1965), Wong (1971), Clark (1966) and after in Clark (1973), as to be shown in Proposition 2.3.29.

**Definition 2.3.27.** Let $(\tau)_{\Delta}$ be a time discretization and $\{W_t\}_{t \in [0,T]}$ be a scalar standard Brownian motion process. **Polygonal approximations** to $W_t$ with differentiable sample paths are defined as follows:

$$W_t^\Delta \triangleq W_{t_n}^\Delta + \frac{W_{t_{n+1}}^\Delta - W_{t_n}^\Delta}{\Delta} (t - t_n), \quad t \in [t_n, t_{n+1}].$$

(2.19)

**Remark 2.3.28.** The simulation of Brownian motion processes provides only values on a finite subset in the time interval $\Delta$ and, if required, these values can be interpolated. Analogously, in the procedure of collecting information about processes being modelled, just measurements over a finite subset on $\Delta$ are performed and again the interpolation can be used to estimate values at other time instants.

According to [Jazwinski (1970) p. 121], such polygonal approximations $W_t^\Delta$ are natural physical approximations to the standard Wiener process $W_t$. Moreover, the polygonal approximation has piecewise differentiable sample paths and, therefore RDE (2.1) with $\xi_t = W_t^\Delta$ has a solution $X_t^\Delta$, i.e. an approximated solution given by the time discretization $(\tau)_{\Delta}$. 

**Proposition 2.3.29.** Let \( f : [0, T) \times \mathbb{R}^D \to \mathbb{R}^D \), \( \sigma : [0, T) \times \mathbb{R}^D \to \mathbb{R} \) be of class \( C^1 \) and \( \sigma \partial \sigma / \partial x \) satisfy a uniform Lipschitz condition in \( x \) and be continuous in \( t \). Furthermore, suppose the initial condition has finite fourth order moment. Then

\[
\lim_{\Delta \to 0} \mathbb{E} \left[ X_t^\Delta \right] = X_t,
\]

where \( \{X_t\}_{t \in [0, T]} \) is the solution of FS SDE (2.4).

**Proof.** See [Wong and Zakai (1965)] and [Clark (1973) Proposition 1]. \( \square \)

As a direct consequence of the proposition above, for modelling plants and/or simulating SDEs by means of analog or even digital computing devices, FS approach provides the most realistic description of the processes [Pugachev, Sinitzyn (1987)]. Then, due to the fact that physical systems are affected by band-limited white noise and computer-based random generators run r.v.s with a small but different from zero correlation interval, the recommendation is always to use FS SDEs. However, this approach has not interesting mathematical properties as the Ito one and its use becomes very restricted. Fortunately, FS and Ito approaches are equivalent for a narrow but important class of systems. In the cases where a process is modelled in Ito sense, the correct way of simulating it in computers is by replacing this mathematical model by its equivalent FS SDE, as described by relation (2.18).

**Remark 2.3.30.** Ito and FS SDE define a diffusion process, though their drift coefficients are in general different.

The importance of FS integral stems from the fact that it models directly the correlations between the random environment and the system. These correlations in the white-noise idealization are a trace of the stochastic dependence between the state of the system and the environmental fluctuations when the latter have a nonvanishing correlation time.
CHAPTER 2. STOCHASTIC MODELLING

For practical calculations, it is convenient to switch over from FS SDE to the equivalent Ito form. The main advantage of Ito version of SDE is the direct displaying of the characteristics of the diffusion process $X_t$ by which a system coupled to an environment with extremely rapid fluctuations has to be modelled. Namely, the drift consists of the phenomenological part $\bar{f}(t, x)$ plus the noise-induced drift $\sigma(t, x)$, where the latter indicates that in a fluctuating environment the systematic motion is modified also if the noise is only approximately white. The diffusion is given by the part of the phenomenological equation that multiplies the fluctuation parameter.

According to Kluen and Platen (1999), FS interpretation of a SDE is the appropriate one when the white noise is used as an idealization of a smooth real noise process. Ito counterpart is a useful artifice which allows access to the appropriate moment equations or the Fokker-Planck equation. In engineering and the physical sciences many SDEs are obtained by including random fluctuations in ODEs, which have been deduced from phenomenological or physical laws. The underlying systems being modelled here are usually continuous in both time and state.

It is much easier to compute expectations of Ito integral than FS integral. As a result, most theoretical work is more conveniently done in Ito framework. In fact, stochastic stability theory (Kushner, 1967b) owes its existence to these properties of Ito integral. Virtually all the theoretical work in stability and control for stochastic systems is done by using Ito integral. Most important of all is the fact that Ito integral is defined for a much broader class of functions when compared to the properties of FS integral. FS integral is so restrictive that it is not applicable to nonlinear filtering theory (Jazwinski, 1970).

2.4 Diffusion Processes

Possibly the most attractive and important property of the solutions of SDEs is that they are usually Markov processes. The idea is to apply the powerful analytical tools that have been developed for Markov and diffusion processes to the solutions of SDEs.
Definition 2.4.1. A stochastic process \( \{X_t\}_{t \in [t_0, T]} \) defined on space \((\Omega, \mathcal{F}, \mathbb{P})\) with index \([t_0, T] \subset [0, \infty)\) and with state space \(\mathbb{R}^D\) is called a Markov process if the following Markov property is satisfied: for \(t_0 \leq s \leq t \leq T\) and all \(B \in \mathcal{B}^D\) (Borel sets in \(\mathbb{R}^D\)), the equation

\[
\mathbb{P}(X_t \in B | \mathcal{F}([t_0, s])) = \mathbb{P}(X_t \in B | X_s) \triangleq \mathbb{P}(s, x; t, B)
\]

holds w.p.1.

Remark 2.4.2. The dynamics of a Markov process is uniquely defined by the infinitesimal generator \(A\).

Definition 2.4.3 (Arnold (1974)). A Markov process \( \{X_t\}_{t \in [t_0, T]} \) with values \(\mathbb{R}^D\) and almost certainly continuous sample functions is called a diffusion process if its transition densities \(p(s, x; t, y)\) satisfy the following conditions for every \(s \in [t_0, T], x \in \mathbb{R}^D\) and \(\epsilon > 0\):

(i) \(\lim_{\epsilon \downarrow 0} \int_{|y-x| > \epsilon} p(s, x; t, y) \, dy = 0\);

(ii) there exists an \(\mathbb{R}^D\)-valued function \(\overline{f}(s, x)\) such that

\[
\lim_{\epsilon \downarrow 0} \int_{|y-x| \leq \epsilon} (y - x)p(s, x; t, y) \, dy = \overline{f}(s, x);
\]

(iii) there exists a \(D \times D\) matrix-valued function \(\sigma(s, x)\) such that

\[
\lim_{\epsilon \downarrow 0} \int_{|y-x| \leq \epsilon} (y - x)(y - x)^T p(s, x; t, y) \, dy = \sigma^2(s, x).
\]

The functions \(\overline{f}\) and \(\sigma\) are well-defined functions and are called the coefficients of the diffusion process. In particular, \(\overline{f}\) is called drift vector and \(\sigma\) is called the diffusion matrix, where \(\sigma\) is symmetric and nonnegative-definite.

In fact, all the conditions in Definition 2.4.3 can be explained as follows:
it prevents a diffusion process from having instantaneous jumps, i.e. large changes in \( X \) over a short period of time are improbable:

\[
P( |X_t - X_s| \leq \epsilon | X_s = x) = 1 - O(t - s);
\]

(ii) it implies that

\[
\overline{f}(s, x) \triangleq \lim_{t \downarrow s} \mathbb{E} [X_t - X_s | X_s = x],
\]

or

\[
\mathbb{E} [X_t - X_s | X_s = x] = \overline{f}(s, x)(t - s) + O(t - s),
\]

so the drift is the instantaneous change rate in the mean of the process given that \( X_s = x; \)

(iii) it means

\[
\sigma^2(s, x) \triangleq \lim_{t \downarrow s} \mathbb{E} [(X_t - X_s)^2 | X_s = x],
\]

or

\[
\mathbb{E} [(X_t - X_s)(X_t - X_s)^T | X_s = x] = \sigma^2(s, x)(t - s) + O(t - s)
\]

\[
= \text{cov}(X_t - X_s | X_s = x),
\]

where \( \text{cov} \) is the covariance matrix of \( X_t - X_s \) with respect to pdf \( p(s, x; t, \cdot) \). The diffusion coefficient denotes the instantaneous change rate of the squared fluctuations of the process given that \( X_s = x. \)

The connection between SDE’s theory and diffusion processes is given by the next theorem.
Theorem 2.4.4. The conditions of Definition 2.3.17 are satisfied by Itô SDE (2.2) with $X_{t_0} = \text{constant}$, $t_0 \leq t \leq T$. If in addition the functions $\mathcal{J}$ and $\sigma$ are continuous with respect to $t$, the solution $X_t$ is a $D$-dimensional diffusion process on $[t_0, T]$.


Remark 2.4.5. In particular, the solution of an autonomous SDE is always a homogeneous diffusion process on $[t_0, \infty)$.

Remark 2.4.6. One can say that the solutions of SDEs and diffusion processes represent essentially the same classes of processes despite their completely different definitions.

In Theorem 2.4.4 it was pointed out that the transition probabilities $\mathbb{P}(s, x; t, B)$ of the solution $X_t$ are equal to the ordinary probability distributions of the solution $X_t(s, x)$ that begins at $x$ at the instant $s$, that is

$$
\mathbb{P}(s, x; t, B) = \mathbb{P}[X_t(s, x) \in B], \ t_0 \leq s \leq t \leq T, \ x \in \mathbb{R}^D, \ B \in \mathcal{B}^D.
$$

Therefore

$$
\mathbb{E}_{s,x} [g(t, X_t)] \triangleq \int_{\mathbb{R}^D} g(t, \gamma) \mathbb{P}(s, x; t, d\gamma) = \mathbb{E} [g(t, X_t(s, x))].
$$

2.4.1 Backward and Forward equations

The main property of diffusion processes is that their transition probability $\mathbb{P}(s, x; t, B)$ is, under certain regularity assumptions, uniquely determined merely by the drift vector and the diffusion matrix.

Let $\{X_t\}_{t \in [t_0, T]}$ denote a $D$-dimensional diffusion process with continuous coefficients $\mathcal{J}(s, x)$ and $\sigma_{ij}(s, x)$. The limit relations in the definition of diffusion process hold uniformly in $s \in [t_0, T]$.
Theorem 2.4.7. Let \( \varphi(x) \) denote a continuous bounded scalar function that

\[
v(s, x) = \mathbb{E} [\varphi(X_t) | X_s = x] = \int_{\mathbb{R}^D} \varphi(y) \mathbb{P}(s, x; t, dy)
\]

for \( s < t \), where \( t \) is fixed, and \( x \in \mathbb{R}^D \) is continuous and bounded, as are its derivatives \( \partial v / \partial x_i \), and \( \partial^2 v / \partial x_i \partial x_j \) for \( 1 \leq i, j \leq D \). Then, \( v(s, x) \) is differentiable with respect to \( s \) and satisfies Kolmogorov’s backward equation

\[
\begin{cases}
\frac{\partial v}{\partial s} + \mathcal{A} v = 0, \\
\lim_{s \uparrow t} v(s, x) = \varphi(x),
\end{cases}
\] (2.21)

where \( \mathcal{A} \) is the operator

\[
\mathcal{A} \triangleq \sum_{i=1}^D f_i(s, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^D g_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j}.
\] (2.22)


Remark 2.4.8. The backward equation (2.22) enables the possibility to determine the transition probability \( \mathbb{P}(s, x; t, \cdot) \). This transition probability is uniquely defined if one knows all the integrals

\[
v(s, x) = \int_{\mathbb{R}^D} \varphi(y) \mathbb{P}(s, x; t, dy),
\]

where \( \varphi \) ranges over a set of bounded functions.

Theorem 2.4.9. If the derivatives \( \partial p / \partial t, \partial (f_i(t, y)p) / \partial y_i \), and \( \partial^2 (\sigma_{ij}(t, y)p) / \partial y_i \partial y_j \) for \( 1 \leq i, j \leq D \) exist and are continuous functions, then for fixed \( s \) and \( x \) such that \( s \leq t \), the transition density \( \mathbb{P}(s, x, t, y) \) is a fundamental solution of Kolmogorov’s forward or Fokker-Planck equation

\[
\begin{cases}
\frac{\partial p}{\partial s} + \mathcal{A}^* p = 0, \\
p(0) = \pi_0,
\end{cases}
\] (2.23)
where $\mathcal{A}^*$ is the formal conjugated operator

$$
\mathcal{A}^* p \triangleq \sum_{i=1}^{D} \frac{\partial}{\partial y_i} (f_i(t, y) p) - \frac{1}{2} \sum_{i,j=1}^{D} \frac{\partial^2}{\partial y_i \partial y_j} (g_{ij}(t, y)p) = 0.
$$

(2.24)

**Proof.** See [Gikhman and Skorokhod (1969)].

**Remark 2.4.10.** The solution $p$ represents the law of development for the state $X_t$ of the stochastic system under consideration. Turning to the law of development for the functions $P(s, x; t, B)$ means to shift from a SDE to a second-order PDE.

### 2.4.1.1 The Feynman-Kac Formula

Let $\alpha$ be a bounded measurable function defined on $\mathbb{R}^D$, and consider the backward PDE

$$
\begin{cases}
\frac{dv}{ds}(s) + \mathcal{A}v(s) + \alpha v(s) = 0, & s \leq t, \\
v(t) = \varphi,
\end{cases}
$$

(2.25)

with $\varphi \in C_0(\mathbb{R}^D)$.

**Theorem 2.4.11.** The Feynman-Kac formula is given $\forall (s, x) \in \mathbb{R}^+ \times \mathbb{R}^D$ by

$$
v(s, x) = \mathbb{E}_x \left[ \varphi(X_t) \exp \int_s^t \alpha(X_\theta) d\theta \right].
$$

(2.26)

The correspondent forward equation is

$$
\begin{cases}
\frac{dp}{ds}(s) = \mathcal{A}^* p(s) + \alpha p(s) = 0, & s \geq 0, \\
p(0) = \pi_0,
\end{cases}
$$

(2.27)

**Theorem 2.4.12.** $\forall (s, x) \in \mathbb{R}^+ \times \mathbb{R}^D$,

$$
\mathbb{E} \left[ \varphi(X_t) Z_t \right] = \int_{\mathbb{R}^D} \mathbb{E}_{0x} \left[ \varphi(X_t) Z_t \right] p_0(x) dx,
$$

(2.28)
where \( \{Z_t\}_{t \geq 0} \) is defined by

\[
dZ_t = \alpha(X_t) Z_t, \quad Z_0 = 1.
\] (2.29)

\[\forall \varphi \in C_0(\mathbb{R}^D).\]

According to [Pardoux (1981)], the relevant generalization of this procedure to nonlinear filtering will consist in replacing \( Z_t \) by an exponential martingale.

## 2.5 Discrete Schemes

In accordance with the definition of the SDEs given by SDEs (2.2) and (2.4), their numerical integration have some peculiarities. Actually, SDEs can be solved by finite approximation schemes in much the same way as ODEs, but their convergence results differ due to the difference between the rules of stochastic and ordinary calculus.

Amongst several well-known discrete approximation methods, strong Taylor approximations are easier to be implemented and provide satisfactory performance benchmarking for comparing solutions of SDEs and ODEs. In this way, assuming the discretization \((\tau)_\Delta\) for a given time interval, the idea is to find a “Taylor-type” expansion \( \{X_t^\Delta\} \) of the exact solution \( \{X_t\} \) about each point of a defined time discretization mesh, and then to approximate terms in this expansion, thereby forming a one-step approximation scheme.

Such an approach gives familiar schemes for ODEs, where the simplest of which is the Euler scheme. An usual performance index of the approximation scheme is the well-known strong convergence criterion ([Kloeden; Platen 1999], p. 323), where the method becomes more accurate when the value of \( \gamma \), the order coefficient, increases.
2.5.1 Strong Convergence and Consistency

**Definition 2.5.1.** Given a time discretization \((\tau)_{\Delta} = \{\tau_n : n = 0, 1, \ldots\}\), a general time discrete approximation \(X^\Delta\) with maximum step size \(\Delta\) converges strongly to \(X\) at time \(T\) if

\[
\lim_{\Delta \downarrow 0} \mathbb{E} \left[ |X_T - X^\Delta_T| \right] = 0.
\]

**Definition 2.5.2.** A time discrete approximation \(X^\Delta\) converges strongly with order \(\gamma > 0\) at time \(T\) if there exists a positive constant \(C\), which does not depend on \(\Delta\), and a \(\Delta_0 > 0\) such that

\[
\epsilon(\Delta) = \mathbb{E} \left[ |X_T - X^\Delta_T| \right] \leq C\Delta^\gamma
\]

for each \(\Delta \in (0, \Delta_0)\).

**Definition 2.5.3 (Kloeden and Platen (1999)).** A discrete time approximation \(X^\Delta\) corresponding to a time discretization \((\tau)_{\Delta}\) with maximum step size \(\Delta\) is **strongly consistent** if there exists a nonnegative function \(c = c(\Delta)\) with

\[
\lim_{\Delta \downarrow 0} c(\Delta) = 0
\]

such that

\[
\mathbb{E} \left\{ \mathbb{E} \left[ \left| \frac{X^\Delta_{n+1} - X^\Delta_n}{\Delta_n} \right| \mathcal{F}_{\tau_n} \right] - f(\tau_n, X^\Delta_n) \right\}^2 \leq c(\Delta) \tag{2.30}
\]

and

\[
\mathbb{E} \left[ \frac{1}{\Delta_n} \left| X^\Delta_{n+1} - X^\Delta_n - \mathbb{E} \left[ X^\Delta_{n+1} - X^\Delta_n \mid \mathcal{F}_{\tau_n} \right] - \sigma(\tau_n, X^\Delta_n) \right|^2 \right] \leq c(\Delta) \tag{2.31}
\]

for all fixed values \(X^\Delta_n = x\) and \(n = 0, 1, \ldots\).
Remark 2.5.4. Condition (2.30) requires the mean of the approximation increment to converge to that of Ito process. From condition (2.31) it follows that the variance of the random parts of the approximation of Ito process also converges to zero. Thus strong consistency gives an indication of the pathwise closeness. In fact, it implies the strong convergence of the time discrete approximation to Ito process.

Theorem 2.5.5 (Kloeden and Platen (1999)). Under the assumptions presented in Definition 2.3.17, for an equidistant time discretization $(\tau)_\Delta$, a strongly approximation $X^\Delta$ of a one-dimensional autonomous Ito process $X_t$ with $X^\Delta_0 = X_0$ converges strongly to $X_t$.

2.5.2 Weak Convergence and Consistency

Definition 2.5.6. A time discrete approximation $X^\Delta$ corresponding to a time discretization $(\tau)_\Delta$ converges weakly to $X$ at time $T$ as $\Delta \downarrow 0$ with respect to a class $\mathcal{C}$ of test functions $g : \mathbb{R}^D \rightarrow \mathbb{R}$ if

$$
\lim_{\Delta \downarrow 0} \left| \mathbb{E} [g(X_T)] - \mathbb{E} [g(X^\Delta_T)] \right| = 0
$$

for all $g \in \mathcal{C}$.

The rate of weak convergence is necessary to compare different time discrete approximations.

Definition 2.5.7. A time discrete approximation $X^\Delta$ converges weakly with order $\beta > 0$ to $X$ at time $T$ as $\Delta \downarrow 0$ if for each $g \in \mathcal{C}_p^{2(\beta+1)}(\mathbb{R}^D, \mathbb{R})$ there exists a positive constant $C$, which does not depend on $\Delta$, and a finite $\Delta_0 > 0$ such that

$$
\left| \mathbb{E} [g(X_T)] - \mathbb{E} [g(X^\Delta_T)] \right| \leq C \Delta^\beta
$$

for each $\Delta \in (0, \Delta_0)$. 
Definition 2.5.8 (Kloeden and Platen (1999)). A discrete time approximation $X^\Delta$ corresponding to a time discretization with maximum step size $\Delta$ is weakly consistent if there exists a nonnegative function $c = c(\Delta)$ with

$$\lim_{\Delta \downarrow 0} c(\Delta) = 0$$

such that

$$\mathbb{E} \left\{ \mathbb{E} \left[ \frac{X_{n+1}^\Delta - X_n^\Delta}{\Delta_n} \bigg| \mathcal{F}_{\tau_n} \right] - \bar{f}(\tau_n, X_n^\Delta) \right\}^2 \leq c(\Delta)$$

(2.32)

and

$$\mathbb{E} \left\{ \mathbb{E} \left[ \frac{1}{\Delta_n} (X_{n+1}^\Delta - X_n^\Delta)(X_{n+1}^\Delta - X_n^\Delta)^T \bigg| \mathcal{F}_{\tau_n} \right] - \sigma(\tau_n, X_n^\Delta) \sigma(\tau_n, X_n^\Delta)^T \right\}^2 \leq c(\Delta)$$

(2.33)

for all fixed values $X_n^\Delta = x$ and $n = 0, 1, \ldots$.

Remark 2.5.9. Condition (2.32) is the same as (2.30). Condition (2.33) is much weaker than (2.31) because only the variance of the approximation increment has to be close to that of Ito process, whereas for strong consistency the variance of the difference between the increments of the approximation and Ito process must vanish.

Under quite natural assumptions a weakly consistent scheme is weakly convergent. The next theorem states an important result.

Theorem 2.5.10 (Kloeden and Platen (1999)). Suppose that $\bar{f}(x)$ and $\sigma(x)$ of an autonomous Ito process are four times continuously differentiable with polynomial growth and uniformly bounded derivatives. Let $X^\Delta$ be a weakly consistent time discrete approximation with equidistant time steps $\Delta_n \overset{\Delta}{=} \Delta$ and initial value $X_0^\Delta = X_0$ which satisfies the moment bounds

$$\mathbb{E} \left[ \max_n \left| X_n^\Delta \right|^{2q} \right] \leq K \left\{ 1 + \mathbb{E} \left[ \left| X_0 \right|^{2q} \right] \right\}$$
for \( q = 1, 2, \ldots \), and

\[
E \left[ \frac{1}{\Delta_n} \left| X_{n+1}^\Delta - X_n^\Delta \right|^q \right] \leq c(\Delta)
\]

for \( q = 1, 2, \ldots \), where \( c(\Delta) \) is as in Definition 2.5.8. Then \( X^\Delta \) converges weakly to the given Ito process.

### 2.5.3 Numerical Stability

Let \((\tau)_\Delta\) be a time discretization and \( n \) be the largest integer satisfying equation (2.7). Let \( X^\Delta \) denote a time discrete approximation with maximum step size \( \Delta > 0 \) starting at time \( t_0 \) at \( X_0^\Delta \) with \( \bar{X}^\Delta \) being the approximation at \( \bar{X}_0^\Delta \).

**Definition 2.5.11.** A time discrete approximation \( X^\Delta \) is **stochastically numerically stable** for a given SDE if for any finite interval \([t_0, T]\) there exists a positive constant \( \Delta_0 \) such that for each \( \epsilon > 0 \) and each \( \Delta \in (0, \Delta_0) \)

\[
\lim_{|X_0^\Delta - \bar{X}_0^\Delta| \to 0} \sup_{0 \leq t \leq T} \mathbb{P} \left( \left| X_{n_t}^\Delta - \bar{X}_{n_t}^\Delta \right| \geq \epsilon \right) = 0
\]

(2.34)

**Remark 2.5.12.** A time discrete approximation is stochastically numerically stable if it is stochastic numerically stable for the class of SDEs for which the approximation converges to the corresponding solution of the equation.

The propagation of an initial error will remain bounded on any bounded interval for a numerically stable scheme. The numerical stability criterion applies only to step sizes \( \Delta > 0 \) that are less than some critical value \( \Delta_0 \), which will usually depend on the time interval \([t_0, T]\) and the differential equation under consideration. This critical value may be extremely small in some cases. For an unstable system, as the time interval becomes large, the propagated error of a numerically stable scheme, becomes so unrealistically large as to make the approximation useless for some practical purposes. In effect it is necessary to control the error propagation over the infinite time interval \([t_0, \infty)\).
Definition 2.5.13. A time discrete approximation $X^\Delta$ is asymptotically stochastically numerically stable for a given SDE if it is numerically stable for any finite interval $[t_0, T]$ and there exists a positive constant $\Delta_0$ such that for each $\epsilon > 0$ and each $\Delta \in (0, \Delta_0)$

$$\lim_{|X^\Delta_n - \hat{X}_n^\Delta| \to 0} \lim_{T \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq T} \left( |X^\Delta_n - \hat{X}_n^\Delta| \geq \epsilon \right) \right) = 0$$

(2.35)

2.5.4 Euler-Maruyama Scheme

One of the simplest time discrete approximations of Ito process is the Euler-Maruyama approximation. Let $(\tau)_\Delta$ be a time discretization and consider the following recursion for the $k$th component of the Euler scheme:

$$X^\Delta_{n+1} = X^\Delta_n + f_n(X^\Delta_n) \Delta + \sigma_n(X^\Delta_n) \Delta W_n, \quad k = 1, 2, \ldots, D. \tag{2.36}$$

Now, as $\Delta \to 0$, the process $\{X^\Delta_n\}$ converges to the solution of Ito SDE (2.2) or, equivalently, to FS SDE (2.3) with the following correction in the drift term:

$$f(t, y) = f(t, y) - \frac{1}{2} \sigma(t, y) \frac{\partial \sigma}{\partial y}(t, y). \tag{2.37}$$

Maruyama (1955) first showed that the Euler scheme converges uniformly in mean-square sense to Ito version of $X_t$ as $\Delta \to 0$. It was shown by Pardoux and Talay (1985) that the order of convergence of the Euler scheme is $\mathbb{E} \left[ X_t - X^\Delta_t \right]^2 = \mathcal{O}(\Delta)$. It represents the simplest strong Taylor approximation and generally attains the order of strong convergence $\gamma = 0.5$ as shown in the next theorem.

Theorem 2.5.14 (Kloeden and Platen (1999)). Suppose the Lipschitz and the initial value conditions in Definition 2.3.17 and that

(i) $\mathbb{E} \left[ |X_0 - X^\Delta_0|^2 \right]^{1/2} \leq K_1 \Delta^{1/2}$,
(ii) \(|\overline{f}(s,x) - \overline{f}(t,x)| + |\sigma(s,x) - \sigma(t,x)| \leq K_2(1 + |x|)|s - t|^{1/2}\),

for all \(s,t \in [0,T]\) and \(x,y \in \mathbb{R}^D\), where the constants \(K_1, K_2\) do not depend on \(\Delta\). Then, for the Euler approximation \(X^\Delta\) the estimate

\[
\mathbb{E} \left[ |X_T - X^\Delta_T| \right] \leq K_3 \Delta^{1/2},
\]

holds, where the constant \(K_3\) does not depend on \(\Delta\).

**Corollary 2.5.15.** When the noise is additive, that is, when the diffusion coefficient has the form \(\sigma(t,x) = \sigma(x)\) for all \((t,x) \in \mathbb{R}^+ \times \mathbb{R}^D\) and under appropriate smoothness assumptions on \(\overline{f}\) and \(\sigma\) it turns that the Euler scheme has order of convergence \(\gamma = 1.0\).

**Remark 2.5.16.** Usually the Euler scheme gives good numerical results when the drift and diffusion coefficients are nearly constant.

In general, the sample path of an Ito process inherits the irregularity of the sample paths of its driving Wiener process, in particular their nondifferentiability. It will not be possible to reproduce the finer structure of such paths on a computer, so the scheme results shall concentrate on the values of a time discrete approximation at the given discretization instants.

### 2.5.5 Milstein Scheme

Milstein scheme can be derived from the Taylor expansion of \(\overline{f}\) and \(\sigma\) when all terms of order \(\Delta\) are kept, remembering that \(\Delta W_n \sim O(\Delta)^{1/2}\). The new formula is represented in FS sense in the multi-dimensional case with \(k = 1, \ldots, D\), where the \(k\)th component of the Milstein scheme is given by

\[
X_{n+1}^{\Delta,k} = X_n^{\Delta,k} + J_n^{k}(X_n^{\Delta,k}) \Delta + \sigma_n^{k}(X_n^{\Delta,k}) \Delta W_n + \frac{1}{2} \sum_{l=1}^{D} \sigma_n^{l}(X_n^{\Delta,k}) \frac{\partial \sigma_n^{k}}{\partial x^l} \left\{ (\Delta W_n)^2 - \Delta \right\},
\]
which means that \( \{X_n^\Delta\} \) converges in the mean square sense to the solution of FS SDE (2.34). The equivalent Ito approach can be obtained by replacing \( \bar{f} \) by \( f \), as presented by relation (2.37).

**Remark 2.5.17.** Milstein scheme ([MILSTEIN 1974]) can also be defined as Euler scheme for FS version of SDE ([PARDOUX; TALAY 1985]).

In other words, to improve the order of convergence to \( \gamma = 1.0 \), the so-called Milstein scheme adds noise-dependent terms to Euler-Maruyama recursion (2.36).

**Remark 2.5.18.** With additive noise the diffusion coefficients depend at most on time \( t \) and not on the \( x \) variable and the Milstein scheme reduces to the Euler scheme, which involves no double stochastic integrals.

**Remark 2.5.19.** For diagonal noise the diagonal diffusion coefficient depends only on \( x^k \), that is, \( \partial \sigma^{ij}/\partial x^k = 0 \) for each \( (t, x) \in \mathbb{R}^+ \times \mathbb{R}^D \) and \( j, k = 1, \ldots, m \) with \( j \neq k \). Thus for diagonal noise the components of Ito process are coupled only through the drift term. Then

\[
X_{n+1}^{\Delta,k} = X_n^{\Delta,k} + \bar{f}_n(X_n^{\Delta,k})\Delta + \sigma_n^k(X_n^{\Delta,k})\Delta W_n + \frac{1}{2} \sigma_n^{k,k}(X_n^{\Delta,k}) \frac{\partial \sigma_n^{k,k}}{\partial x^k} \{(\Delta W_n)^2 - \Delta\}. \tag{2.38}
\]

An important result associated with the Milstein scheme can be stated in the next theorem as follows ([MILSTEIN 1974]–[PARDOUX; TALAY 1985]).

**Theorem 2.5.20.** For the diagonal case, suppose the set of assumptions presented in Theorem 2.5.14. Then, for Milstein approximation \( X^\Delta \) the estimate

\[
\mathbb{E} \left[ |X_T - X_T^\Delta| \right] \leq K_3 \Delta,
\]

holds, where the constant \( K_3 \) does not depend on \( \Delta \).

**Remark 2.5.21.** This second order scheme is the proper generalization of the deterministic Euler scheme for the strong convergence criterion because it gives the same order of convergence as for the deterministic case, that is, \( \gamma = 1.0 \).
In many important practical problems the diffusion coefficients have special properties which allow the Milstein scheme to be simplified in a way that avoids the use of double stochastic integrals involving different components of the Wiener process \cite{KLOEDEN:1999}.

## 2.6 Examples

This section illustrates the necessary mathematical care in the modelling of plants and in simulating their associated SDEs in computers \cite{JACOB:2006}, a contribution of this thesis recalling an important result from the 1970’s.

### 2.6.1 SDE with Wiener Integral

Let $\sigma : [0, \infty) \to \mathbb{R}$ be a deterministic function and

\[ dX_t = f(t, X_t)dt + \underbrace{\sigma(t) \circ dW_t}_{\text{Wiener Integral}} \]  \hspace{1cm} (2.39)

be a FS SDE modelling a physical system. Based on relation (2.18), the equivalent Ito SDE can be written in the same way, \textit{i.e.} $f(t, x) = \overline{f}(t, x)$, once the added correction term in the given relation does not depend on $x$.

Furthermore, assuming the discretization $(\tau)_\Delta$, equation (2.39) can be approximated by the Milstein scheme, as presented below:

\[ X^\Delta_{n+1} = X^\Delta_n + f_n(X^\Delta_n)\Delta_n + \underbrace{\sigma_n \Delta W_n}_{\equiv \xi_n}, \]

where, for random number generators in computers, $\xi_n$ are $N(\sigma_n, \Delta_n)$ normally distributed random variables with correlation $\delta_0 \to 0$, \textit{i.e.} the traditional white noise sequence. In relation (2.37) the drift has no correction, \textit{i.e.} $f(n, y) = \overline{f}(n, y)$, then the Milstein and the Euler-Maruyama approaches are equivalent for systems with Wiener
integrals. It means that the order of convergence $\gamma = 1.0$ is performed by both methods and it is equivalent to that obtained from the Euler method for ODEs.

The results obtained by SDEs with Wiener integrals are equivalent to the approach which first discretizes the deterministic state-space system and after includes the white noise disturbances, as suggested by Aström (1970) in the modelling of physical systems.

### 2.6.2 General Case

This case represents a plant modelled as FS SDE (2.4), where the equivalent Ito SDE can be described by the process $\{U_t\}_{t \in [0,T]}$ defined as

$$U_t = X_t + \frac{1}{2} \int_0^T \sigma(t, X_t) \frac{\partial \sigma}{\partial x}(t, X_t) dt,$$

(2.40)

an immediate consequence of Proposition 2.3. This result shows that the processes $\{X_t\}$ and $\{U_t\}$ are different, which means FS and Ito modelling approaches differ from a drift term which depends on the state. However, if $\sigma(t, x)$ belongs to a class of smooth functions in relation to $x$, then $\partial \sigma / \partial x$ is ‘small’, what implies that $X_t$ can be a good approximation of $U_t$ in equation (2.40).

For the approximation schemes with discretization $(\tau)_\Delta$, the Milstein scheme can be rewritten as a function of the Euler-Maruyama one by defining the process $\{U_n^\Delta\}$ given in FS sense by

$$U_{n+1}^\Delta = X_{n+1}^\Delta + \frac{1}{2} \sigma_n(X_n^\Delta) \frac{\partial \sigma_n}{\partial x} \{(\Delta W_n)^2 + \Delta\}. $$

Now, the terms which improve the convergence order belong to the drift and diffusion terms of the given SDE, but both still depend on the existence of the state variable $x$ and on the diffusion function $\sigma$. Assuming, again, that $\sigma$ is smooth, both approximation schemes become closer. The same argument should be used if $\{U_n^\Delta\}$ were defined in Ito sense.
3 Continuous-Time Nonlinear Filtering

3.1 Introduction

The most natural model for filtering signals is that of an $\mathbb{R}^M$-valued observation $y_t$ given by

$$y_t = \hat{h}(t, x_t) + \zeta_t,$$

where the function $\hat{h}(t, x)$ provides the measured signal and $\zeta_t$ represents the uncertainty in the measurement procedure. This signal contains information about the states $x_t$ of a physical system whose values need to be known. In precise terms, the process representing the states, now $X$, is called the signal process and is usually a Markov process taking values in $\mathbb{R}^D$. For each time instant $t$, the function $\hat{h}$ is measurable with respect to the filtration $\mathcal{F}_t^X$ which is the $\sigma$-field generated by the family $\{X_s\}_{s \in [0,t]}$, augmented by the inclusion of all zero probability sets. The process $\zeta_t$ is, intuitively speaking, Gaussian white noise. To make the measurement model (3.1) rigorous it is necessary to consider the integrated version

$$Y_t = \int_0^t h(s, X_s) \, ds + V_t, \quad 0 \leq t \leq T,$$

where $\{V_t\}$ is not necessarily the standard Wiener process and the function $h$ can be directly mapped from $\hat{h}$. An important technical assumption is the independence of signal and noise, i.e. $\sigma\{X_t, 0 \leq t \leq T\}$ and $\sigma\{W_t, 0 \leq t \leq T\}$ are independent. It will
also be assumed that $\int_0^T |h_t|^2 dt < \infty$, with suitable measurability assumptions on $h$.

A fundamental criticism of filtering theory based on stochastic calculus is that the conventional filtering model (3.2) in which the noise is modelled by a Wiener process has no practical validity because, from the standpoint of applications, the results obtained cannot be instrumented (Kallianpur; Karandikar 1983). The investigations on pathwise or robust solutions for filtering (Clark 1978)–(Davis 1979) as well as the work of recent years to obtain approximations to Itô SDEs and their solutions may be viewed as an attempt to bring the conventional theory to applications, as described by the books of Ikeda and Watanabe (1981), Kloeden and Platen (1999), Del Moral (2004) and Milstein and Tretyakov (2004).

The basic motivation of the nonlinear filtering problem is the following: given the observations $Y = \{Y_s\}_{s \in [0,t]}$ and a parametric model describing the physical one, finding a conditional distribution of the “present” state of the system. Moreover, since in most technological problems, the data is not observed continuously over time, it is of practical importance to obtain a recursive estimate so that the conditional distribution at time $t + \Delta$, $\Delta > 0$, can be computed using its value at time $t$. Hence one looks for a SDE for the conditional distribution or, equivalently, for the conditional expectation $\pi_t(\varphi) = E[\varphi(X_t)|\mathcal{F}_t]$ for a rich enough class of functions $\varphi: \mathbb{R}^D \rightarrow \mathbb{R}$, $D \geq 1$. It is sufficient to consider $\varphi \in C_b^2$.

There are essentially two different approaches to the nonlinear filtering problem. The first is based on the important idea of innovations processes, originally introduced by Hendrik W. Bode and Claude E. Shannon in the context of Wiener filtering problems and later developed by Thomas Kailath and his students in the late 1960’s for nonlinear filtering problems. This approach reaches its culmination in the seminal paper of Fu-jisaki, Kallianpur and Kunita (1972). A detailed account of this approach is available in the books of Lipster and Shiryayev (1977) and Kallianpur (1980). The second approach can be traced back to the doctoral dissertations of Mortensen (1966) and Duncan (1964), and the important paper of Zakai (1969). In this approach attention is focused on the
unnormalized conditional density equation, which is a bilinear PDE, and it derives its inspiration from function space integration as originally introduced by [Kac, 1951]. Mathematically, this view is closely connected to the path integral formulation obtained in the Quantum Physics (Feynman; Hibbs, 1965).

The case of interest in this thesis occurs when the signal process \( \{X_t\} \) and the observation noise \( \{V_t\} \) in model (3.2) are independent. The conditional expectation \( \pi_t(\varphi) = \mathbb{E}[\varphi(X_t) | \mathcal{F}_t^V] \) for this case can be computed via the generalized Bayes formula, the so-called Kallianpur-Striebel (KS) formula (Kallianpur; Striebel, 1968). The advantage of this approach is that the solution can be obtained by the SPDE called Zakai equation (Zakai, 1969), where an optimal filter can be derived directly from KS formula without having to use sophisticated martingale theoretic arguments (Kailath, 1974), as those presented by Fujisaki, Kallianpur and Kunita (1972).

The stochastic filtering problem consists in effectively estimating the conditional distribution of a process given the noisy information obtained from a related process. The basic problem can be identified in many applications: signal processing, radar control, satellite tracking, weather forecasting, speech recognition. Except for a small number of cases, the problem does not admit a solution in closed form. Therefore, efficient numerical approximations to the conditional distribution are of great interest. In this way, the possibility of rewriting the results in accordance with the representation of Feynman-Kac formula (Del Moral, 2004) is the basic motivation to use MC methods to solve numerically the filtering problem.

Next sections present the main results of the classical nonlinear filtering problem. Motivated by applications in aerospace systems, special focus is given over the deduction of the robust representation when the observation model contains Brownian motion with a diagonal time-based covariance matrix, a contribution of this thesis.
3.2 Basic Framework

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space equipped with a filtration \(\mathcal{F}_{t \geq 0}\) on which is given, for \(D, M \geq 1\), a \((D + M)\)-dimensional standard Brownian motion \(((W_t, V_t), \mathcal{F}_t)_{t \geq 0}\) independent of \(\pi_0\), a \(D\)-dimensional \(\mathcal{F}_0\)-measurable random vector with

\[
\mathbb{E} \left[ |\pi_0|^2 \right] < \infty, \quad \text{i.e.} \quad \int |x|^2 d\mathbb{P}_0(x) < \infty,
\]

where \(d\mathbb{P}_0(x) = q_0(x) \, dx\) is a given probability measure.

3.2.1 Signal Process

The so-called signal process \(\{X_t, \mathcal{F}_t\}_{t \geq 0}\) is the following unobserved Markov diffusion process represented by Ito’s SDE

\[
\begin{cases}
    dX_t = \overline{f}(t, X_t) \, dt + \sigma(t, X_t) \, dW_t, \\
    X_0 \sim \pi_0,
\end{cases}
\tag{3.3}
\]

where \(\overline{f} : \mathbb{R}_+ \times \mathbb{R}^D \rightarrow \mathbb{R}^D\) and \(\sigma : \mathbb{R}_+ \times \mathbb{R}^D \rightarrow \mathcal{L}(\mathbb{R}^D, \mathbb{R}^D) \simeq \mathbb{R}^{D^2}\) are globally Lipschitz functions. In accordance with the classical theory of SDEs in Ito sense [Liptser, Shiryayev (1977)], these hypothesis guarantee the existence and uniqueness of the solution of SDE (3.3) and

\[
x(\cdot) \in L^2 \left( \Omega, \mathcal{F}, \mathbb{P}; C \left( [0, T]; \mathbb{R}^D \right) \right),
\]

for any fixed final time \(T > 0\), where from now on the small case letter \(x\) represents the realizations of the process \(X\).
3.2.2 Observation Process

The generalized observation process \( \{Y_t, \mathcal{F}_t\}_{t \geq 0} \) satisfies the following Markov process represented by the equation

\[
\begin{cases}
    dY_t = h(t, X_t) \, dt + dV_t, \\
    Y_0 = 0,
\end{cases}
\]

(3.4)

where \( h : \mathbb{R}_+ \times \mathbb{R}^D \to \mathbb{R}^M \) is globally Lipschitz and the Wiener process \( \{V_t\}_{t \geq 0} \) has the uniformly positive definite matrix \( R(t) \), a time-dependent diagonal covariance matrix.

**Remark 3.2.1.** The linear growth bound of the function \( h \) imposed by the Lipschitz conditions implies that (BENSOUSAN, 1992)

\[
\mathbb{E} \left[ \int_0^T |h(s, X_s)|^2 \, ds \right] < \infty.
\]

3.3 The Filtering Problem

The traditional filtering problem within the frame \([0, T]\) consists in determining the conditional law of the signal \( X \) given all the observational information represented by the filtration \( \mathcal{Y}_t \triangleq \sigma(Y_s, 0 \leq s \leq t) \subset \mathcal{F}_t \). In other words, the problem consists in evaluating for all test function \( \varphi \in \mathcal{C}_b \) the conditional expectation

\[
\pi_t(\varphi) \triangleq \mathbb{E} [\varphi(X_t) | \mathcal{Y}_t], \quad \mathbb{P} - \text{a.s.},
\]

(3.5)

for all \( t \in [0, T] \), so that

\[
\begin{cases}
    \pi_t(\varphi) = \int_{\mathbb{R}^D} \varphi(x) \, \pi_t(dx), \\
    \pi_0(dx) = d\mathbb{P}_0(x).
\end{cases}
\]

(3.6)

The main core of the theory consists in characterizing \( \pi_t(\varphi) \) as the solution of an
evolution equation. It is desired this computation to be done recursively in terms of a given statistic $\gamma$ which can be updated using only new observations and from which estimates can be calculated in a pointwise or memoryless fashion (Davis; Marcus, 1981), i.e.

$$\pi_{t+\tau} = \gamma(t, \tau, \pi_t, \{Y_{t+s}, 0 \leq s \leq \tau\}), \quad \tau \leq T.$$ 

**Remark 3.3.1.** The conditional expectation $\theta(Y_t) \overset{\Delta}{=} \mathbb{E} [\varphi(X_t)|Y_t]$ is the optimal estimate which minimizes the mean square error $\mathbb{E} [\|\varphi(X_t) - \theta(Y_t)\|^2]$ (Van Trees, 1968), where $\varphi$ is adequately chosen to compute the moments of the signal process $X$ at time $t$.

### 3.3.1 Change of Probability Measure

It will be convenient to modify the measure of probability on $\Omega$ in order to transform the observation process $Y_t$ into the standard Wiener process $B_t$. For that it is introduced the process $\eta_t$ defined by

$$\begin{cases} 
\, d\eta_t = -\eta_t h^*(t, X_t) R^{-1}(t) dB_t, \\
\, \eta_0 = 1,
\end{cases}$$

or explicitly as

$$\eta_t = \exp \left\{ -\int_0^t h^*(s, X_s) R^{-1}(s) dB_s - \frac{1}{2} \int_0^t h^*(s, X_s) R^{-1}(s) h(s, X_s) ds \right\}. \quad (3.7)$$

**Lemma 3.3.2.** The martingale exponential (3.7) has (Girsanov, 1960)

$$\mathbb{E} [\eta_t] = 1.$$ 

By virtue of Lemma 3.3.2 the so-called Girsanov theorem (Liptser; Shiryaev, 1977)
defines a new probability \( \hat{\mathbb{P}} \) by setting the Radon-Nikodym derivative
\[
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}|_{Y_t} = \eta_t, \tag{3.8}
\]
where, under this new measure \( \hat{\mathbb{P}}(d\omega) = \eta \mathbb{P}(d\omega) \), the observation \( Y_t \) becomes a Wiener process independent of the signal \( X_t \).

**Remark 3.3.3.** In other mathematical notation, let \( \mathbb{P}_{sx} \) be the law of the solution of the signal \( X_t \) with initial condition \( X_0 = x \), \( \theta \leq s \), and \( V_{sx} \) be a Wiener measure on \( \mathcal{C} \left( [s, \infty); \mathbb{R}^D \right) \) starting at time 0. Then, under
\[
\hat{\mathbb{P}}_{sx}(dX, dY) \triangleq \mathbb{P}_{sx}(dX)xW_0(dY),
\]
the observation \( Y_t \) is a standard Wiener process independent of \( X_t \).

**Definition 3.3.4.** The so-called Girsanov exponential means the likelihood ratio given by
\[
\frac{1}{\eta_t} = \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}|_{Y_t} = \Lambda_t, \tag{3.9}
\]
or explicitly as
\[
\Lambda_t = \exp \left\{ \int_0^t h^*(s, X_s)R^{-1}(s)dB_s + \frac{1}{2} \int_0^t h^*(s, X_s)R^{-1}(s)h(s, X_s)ds \right\} = \exp \left\{ \int_0^t h^*(s, X_s)R^{-1}(s)dY_s - \frac{1}{2} \int_0^t h^*(s, X_s)R^{-1}(s)h(s, X_s)ds \right\}. \tag{3.10}
\]

**Remark 3.3.5.** It follows from Radon-Nikodym derivative (3.8) that
\[
\hat{\mathbb{E}}[\eta_t] = \mathbb{E}[\Lambda_t \eta_t] = 1.
\]

Then it is possible to express the conditional expectation \( \pi_t(\varphi) \) by KS formula (Kallianpur; Striebel, 1968), though special versions occur in the papers of Bucy (1965), Won-
Lemma 3.3.6 (Kallianpur–Striebel formula). The following formula holds:

\[ \pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)}, \quad \mathbb{P} - \text{a.s.,} \quad (3.11) \]

where

\[ \rho_t(\varphi) = \tilde{E}[\varphi(X_t)\Lambda_t|\mathcal{Y}_t] \quad (3.12) \]

is the so-called unnormalized conditional distribution on \( X_t \).

Remark 3.3.7. Defining \( \mathbb{P} \) corresponding to the initial density \( q_0 \) at time 0 and realizing that \( X_t \) and \( Y_t \) are independent processes then follows:

\[ \tilde{E}_{sx}[\varphi(X_t)\Lambda_s|\mathcal{Y}_s] = E_{sx}[\varphi(X_t)\Lambda_s], \quad W_{s0}-\text{a.s.,} \]

where the operator \( \tilde{E} \) means the expectation with respect to the new transformed measure \( \tilde{\mathbb{P}} \).

3.3.2 Zakai Equation

For practical purposes it would be very interesting if the solution of the nonlinear filtering problem had also a recursive form, instead of the one provided by KS formula (3.11) giving \( \pi_t(\varphi) \) only non-recursively as a function-space integral. This aim was reached with the following equation called Zakai equation (ZAKAI 1969) via the use of a SPDE, though many improvements occur thereafter in relation to the framework required for the existence and uniqueness of the mathematical solution, as presented by Pardoux (1979) and Bhatt, Kallianpur and Karandikar (1995).
Theorem 3.3.8 (Zakai equation). Let \( \varphi(x) \in C^{2,1}_b(\mathbb{R}^D) \), then one has the relation

\[
d\rho_t(\varphi) = \pi_0(\varphi) + \int_0^t \rho_s \left( \frac{\partial \varphi}{\partial s} - A_s \varphi \right) ds + \int_0^t \rho_s (h^*(s, X_s) \varphi) R^{-1}(s) dY_s, \ \mathbb{P}\text{-a.s.}, \tag{3.13}
\]

for all \( t \), where the so-called infinitesimal generator \( A \) is given by the following second order differential operator

\[
A_t \varphi \triangleq \frac{1}{2} \sum_{i,j=1}^D g_{ij}(s, x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^D f_i(s, x) \frac{\partial \varphi}{\partial x_i},
\]

(3.14)

\[
= \frac{1}{2} \text{tr} [G(s, x) \nabla (\nabla \varphi)] + \tilde{f}(s, x) \nabla \varphi, \tag{3.15}
\]

with each element of the noise covariance matrix \( G(t, x) \) representing the drift of the signal process \( X_t \) defined as

\[
g_{ij}(t, x) = \sigma_i(t, x)\sigma_j(t, x), \quad i, j = 1, \ldots, D. \tag{3.16}
\]

Remark 3.3.9. The stochastic integral

\[
\int_0^t \rho_s (h^*(s, X_s) \varphi) R^{-1}(s) dY_s
\]

is backward solved. Therefore, Zakai equation (3.13) is a backward SPDE.

For application purposes in solving the filtering problem, equation (3.13) can be friendly rewritten as a forward SPDE.

Corollary 3.3.10. Let \( \varphi(x) \in C^{2,1}_b([\mathbb{R}^D]), q(t, x) \) is the unnormalized density of the conditional law of \( X_t \) given \( \mathcal{Y}_t \), that is (PAUROSUX, 1981)

\[
\int_{\mathbb{R}^D} q_o(x) \mathbb{E}_{0x} [\varphi(X_t) \Lambda_t | \mathcal{Y}_t] dx = \mathbb{E} [\varphi(X_t) \Lambda_t | \mathcal{Y}_t], \ \forall t \geq 0,
\]
where \( q_t(x) = q(t, x) \) is solution of the forward SPDE

\[
q_t(x) = q_0(x) + \int_0^t A^*_s q_s(x) \, ds + \int_0^t (h^*(s, X_s) R^{-1}(s) q_s(x) \, dY_s, \, \mathbb{P}\text{-a.s.},
\]

with \( A^* \) being the formal adjoint given by the following differential operator

\[
A^*_i \varphi \triangleq -\frac{1}{2} \sum_{i,j=1}^D \frac{\partial^2}{\partial x_i \partial x_j} (g_{ij}(s, x) \varphi) + \sum_{i=1}^D \frac{\partial}{\partial x_i} \left( f^i(s, x) \varphi \right)
\]

\[
= -\frac{1}{2} \text{tr} \left( \nabla G(s, x) \varphi \right) + \nabla \left( f(s, x) \varphi \right).
\]

### 3.3.3 Feynman-Kac Representation

KS formula (3.11) gives \( \pi_t(\varphi) \) non-recursively as a function-space integral. Combining equations (3.11) and (3.8) and using the conditions imposed in Zakai equation (3.13), the solution of the nonlinear filtering problem can be rewritten in terms of Feynman-Kac formula (DELMORAL, 2004)

\[
\pi_t(\varphi) = \frac{\int_{\mathbb{R}^D} \varphi(x') e^{U(t, x', dy_t)} \mathbb{P}(dx')}{\int_{\mathbb{R}^D} e^{U(t, x', dy_t)} \mathbb{P}(dx')},
\]

where the potential (FEYNMAN; HIBBS, 1965) is defined as

\[
U(t, x_t, dy_t) \triangleq \int_0^t h^*(s, x_s) R^{-1}(s) \, dy_s - \frac{1}{2} \int_0^t h^*(s, x_s) R^{-1}(s) h(s, x_s) \, ds.
\]

Therefore, the nonlinear filtering problem can be solved if it is possible to compute the measure \( \mathbb{P}(dx) = p(t, x') \, dx' \), where \( p(t, x) \) is solution of the so-called Fokker-Planck equation

\[
\frac{\partial p(s, x)}{\partial s} + A^* p(s, x) = 0, \quad p(0, x) \sim \pi_0.
\]
3.4 Robust Representation

Clark (1978) and Davis (1979) showed that the solution of the nonlinear filtering formula (3.11) can be obtained separately for each sample path of the observation \( Y_t \). The reason why this is important is explained clearly by Kushner (1979) and it has to do with the question of stochastic modelling.

**Definition 3.4.1 (Clark (1978)).** For \( \hat{\varphi} : C_{R^M}[0, t] \rightarrow R \), a robust representation of the conditional expectation \( \pi_t(\varphi) \) presented by (3.11) is of the form

\[
\pi_t(\varphi) = \hat{\varphi}(Y_t) \quad \mathbb{P}\text{-a.s.,}
\]

where \( Y_t \) is a path-valued r.v.

The need for this type of representation arises when the filtering framework is used to model and solve real-life problems (Clark; Crisan 2005). The observation model \( Y \) chosen for the real observation process \( \hat{Y} \) may not be a perfect one, e.g. the additive uncertainty is a physical wide band noise. However, as long as the distribution of \( \hat{Y} \) is close in a weak sense to that of \( Y_t \), the estimate \( \hat{\varphi}(\hat{Y}_t) \) computed on the actual observation is still reasonable, as \( \mathbb{E} \left[ (\varphi - \hat{\varphi}(Y_t))^2 \right] \) is well approximated by the idealized error \( \mathbb{E} \left[ (\varphi - \hat{\varphi}(\hat{Y}_t))^2 \right] \). In this way, the search for a particular version of the conditional expectation (3.11) which has nice properties is of paramount importance for the application of the nonlinear filtering theory in practice.

Even in cases where \( Y_t \) and \( \hat{Y}_t \) coincide, one is never able to obtain and exploit a continuous stream of data as modelled by the continuous path \( Y_t(\omega) \). Instead the observation arrives and is processed at discrete moments given by the time discretization \( (\tau)_{\Delta} \). However the continuous path \( Y_t(\omega) \) obtained from the discrete observations \( (Y_{\tau_i}^\Delta(\omega))_{i=0}^N \) by linear interpolation is close to \( Y_t(\omega) \) (with respect to the supremum norm on \( C_{R^M}[0, t] \)); hence, by the same argument, \( \hat{\varphi}(\hat{Y}_t) \) will be a sensible approximation of \( \pi_t(\varphi) \).
Let $\Theta(y_\cdot)$ be the following r.v.:

$$
\Theta(y_\cdot) \triangleq \exp \left\{ H(t, X_t)y_t - I(y_\cdot) - \frac{1}{2} \int_0^t H(s, X_s)^T H(s, X_s) ds \right\},
$$

where $I(y_\cdot)$ is a certain version of the stochastic integral $\int_0^t y_s^T dH_s$. The exponent of $\Theta(y_\cdot)$ will be recognized as a formal integration-by-parts of the exponent in KS formula given in equation (3.11).

Finally, let $\hat{g}^\phi, \hat{g}^1, \hat{\varphi} : C_{\mathbb{R}^D}[0, t] \to \mathbb{R}$ be the following functions

$$
\hat{g}^\phi(Y_\cdot) = \hat{E}[\phi(X_\cdot, Y_\cdot)\Theta(Y_\cdot)], \quad \hat{g}^1 = \hat{E}[\Theta(Y_\cdot)], \quad \hat{\varphi}(Y_\cdot) = \frac{\hat{g}^\phi(Y_\cdot)}{\hat{g}^1(Y_\cdot)},
$$

where $\hat{\varphi}$ is the candidate for the robust form of $\pi_t(\varphi)$. More precisely, the next two theorems deduced by Clark and Crisan (2005) are the end of the work initiated by Clark (1978) and Kushner (1979).

**Theorem 3.4.2.** The function $\hat{\varphi}$ is locally Lipschitz. In other words, for any $C > 0$, there exists a constant $K_C$ such that

$$
|\hat{\varphi}(Y^1) - \hat{\varphi}(Y^2)| \leq K_C ||Y^1 - Y^2||
$$

for any two paths $Y^1, Y^2$ such that $||Y^1||, ||Y^2|| \leq C$.

**Theorem 3.4.3.** The r.v. $\hat{\varphi}$ is a version of $\pi_t(\varphi)$; that is $\pi_t(\varphi) = \hat{\varphi}(Y_\cdot), \mathbb{P}$-a.s. Hence $\hat{\varphi}(Y_\cdot)$ is the unique robust representation of $\pi_t(\varphi)$.

Next section presents a transformation expressing the conditional expectation $\pi_t(\varphi)$ in its robust representation and in terms of solutions of PDEs. This transformation was first introduced in the references of Doss (1977) and Sussmann (1978), but the result was only interpreted in terms of robustness by Lipster and Shiryaev (1977) and Clark (1978).
3.4.1 Pathwise Filtering

The $k$th component of the signal in (3.3) is given by

$$dX^k_t = \int f^k_t \, dt + \sum_{i=1}^D \sigma^{k,i}_t \, dW^k_t, \quad k = 1, 2, \ldots, D,$$

(3.26)

with $\int f^k_t \triangleq f^k(t, x)$, $\sigma^{k,i}_t \triangleq \sigma^k(t, x)$.

Motivated by the modelling of physical systems in the aerospace field [MERHAV 1996], (MALYSHEV et al. 1996), the $m$th component of the observation model in (3.4) is defined as follows:

$$dY^m_t = h^m_t \, dt + \lambda^m_t \, dV^m_t, \quad m = 1, 2, \ldots, M,$$

(3.27)

with $h^m_t \triangleq h^m(t, x)$ and $\lambda^m_t \triangleq \lambda^m(t)$ being the root of the correspondent diagonal elements of the covariance matrix $R(t)$.

The associated Zakai equation is now given by the following set of SPDEs

$$\begin{cases}
    d\rho(s) + A_s \rho(s) \, ds + \sum_{m=1}^M h^m_s \lambda^m_s \, dY^m_s = 0, \quad s < t,
    \\
    \rho(t) = \varphi \ (\text{given}), \quad s = t;
\end{cases}$$

and

$$\begin{cases}
    dq(s) = A_s^* q(s) \, ds + \sum_{m=1}^M h^m_s \lambda^m_s \, dY^m_s, \quad s > 0,
    \\
    q(0) = q_0 = \pi_0, \quad s = 0.
\end{cases}$$

Since the operators multiplication by $h^m$ and multiplication by $h^j$ commute, it is possible to use the device of Doss (1977) and Sussmann (1978) to express the solutions in terms of solutions of ordinary PDE’s. The idea is to find equations for the quantities: $u(t, x) = \rho(t, x) \exp[Y_t h(t, x)]$ and $\bar{u}(t, x) = q(t, x) \exp[-Y_t h(t, x)]$, instead of $\rho$ and $q$ in Zakai equation. This has been done indeed by Lipster and Shiryaev (1977) and Clark
(1978) who interpreted this result in terms of robustness. See the discussion in the work of Mitter (1982).

This direct derivation of the robust equations for $u$ and $\overline{u}$ has two interesting features (Pardoux, 1981): 1-) it is rather elementary because uses only theory of PDE, Girsanov transformation, and Feynman-Kac formula, and 2-) it does work well when $f$ and $\sigma$ depend on the whole past of $Y$. These ideas are closely related to those from Davis (1981b).

### 3.4.2 Integration by Parts

The Girsanov exponential (3.10) can be rewritten componentwise as

$$
\Lambda_t^s = \exp \left( \int_s^t \sum_{m=1}^D \frac{1}{\lambda^m} h_r^m Y^m - \frac{1}{2} \int_s^t \sum_{m=1}^D \frac{1}{\lambda^m} (h_r^m)^2 dr \right),
$$

where $\lambda^m > 0$ for all $t$.

For integrating by parts the Girsanov exponential (3.28) it is necessary first the definition of basic terms and the development of some important results.

**Definition 3.4.4.**

$$
\sum_{m=1}^M \frac{1}{\lambda^m} h_r^m Y^m \triangleq \sum_{m=1}^M \bar{Y}_r^m h_r^m \triangleq (\bar{Y}_r, h_r),
$$

where

$$
\frac{1}{\lambda^m} Y^m \triangleq \bar{Y}_r^m
$$

and $(\cdot, \cdot)$ denotes a scalar product.

**Definition 3.4.5.**

$$
\frac{1}{\lambda^m} h_r^m \triangleq \bar{h}_r^m.
$$
Applying the last two definitions into equation (3.28), a compact notation of the Girsanov exponential is given by

\[
\Lambda_t^s = \exp \left( \int_s^t \sum_{m=1}^M \tilde{h}_r^m dY_r^m - \frac{1}{2} \int_s^t \sum_{m=1}^M \tilde{h}_r^m h_r^m \, dr \right)
\]

\[
= \exp \left( \int_s^t (\tilde{h}_r, dY_r) - \frac{1}{2} \int_s^t (\tilde{h}_r, h_r) \, dr \right).
\] (3.29)

**Proposition 3.4.6.** The differential term in the first integral of the exponential equation (3.29) is given by

\[
d \left( \tilde{Y}_r, h_r \right) = \left( \tilde{Y}_r, dh_r \right) + (\tilde{h}_r, dY_r) - (\tilde{Y}_r, \tilde{h}_r) \, dr.
\] (3.30)

**Proof.** Using ideas described by Pardoux (1981), Pardoux (1982) and Souza (1992), the differential can be rewritten as

\[
d \left( \tilde{Y}_r, h_r \right) = d \left( \sum_{m=1}^M \tilde{Y}_r^m h_r^m \right) = \sum_{m=1}^M d \left( \tilde{Y}_r^m h_r^m \right),
\]

but from Itô sense

\[
d \left( \tilde{Y}_r^m h_r^m \right) = \tilde{Y}_r^m dh_r^m + h_r^m d\tilde{Y}_r^m + d < \tilde{Y}_r^m, h_r^m >
\]

\[
= \tilde{Y}_r^m dh_r^m + h_r^m d\tilde{Y}_r^m
\]

because \( \tilde{Y}_r^m \) and \( h_r^m \) are independent for all \( m \). Then \( d < \tilde{Y}_r^m, h_r^m > = 0 \).

Therefore,

\[
d \left( \tilde{Y}_r, h_r \right) = \sum_{m=1}^M \tilde{Y}_r^m dh_r^m + \sum_{m=1}^M h_r^m d\tilde{Y}_r^m = \left( \tilde{Y}_r, dh_r \right) + (h_r, d\tilde{Y}_r).
\]

Now, from the classical calculus

\[
d\tilde{Y}_r^m = d \left( \frac{1}{\lambda_r^m} Y_r^m \right) = \frac{1}{\lambda_r^m} dY_r^m - \frac{1}{(\lambda_r^m)^2} Y_r^m \, dr,
\]
then

\[
(h_r, d\bar{Y}_r) = \sum_{m=1}^{M} \frac{1}{\lambda_r^m} h_r^m dY_r^m - \sum_{m=1}^{M} \frac{1}{(\lambda_r^m)^2} Y_r^m h_r^m dr
\]

\[
= \sum_{m=1}^{M} \bar{h}_r^m dY_r^m - \sum_{m=1}^{M} Y_r^m \bar{h}_r^m dr
\]

\[
= (\bar{h}_r, dY_r) - (\bar{Y}_r, \bar{h}_r) dr.
\]

Finally

\[
d(\bar{Y}_r, h_r) = \sum_{m=1}^{M} \bar{Y}_r^m dh_r^m + \sum_{m=1}^{M} \bar{h}_r^m dY_r^m - \sum_{m=1}^{M} \bar{Y}_r^m \bar{h}_r^m dr
\]

\[
= (\bar{Y}_r, dh_r) + (\bar{h}_r, dY_r) - (\bar{Y}_r, \bar{h}_r) dr.
\]

By developing (3.30) in the integral form

\[
(\bar{Y}_t, h_t) - (\bar{Y}_s, h_s) = \int_s^t (\bar{Y}_r, dh_r) + \int_s^t (\bar{h}_r, dY_r) - \int_s^t (\bar{Y}_r, \bar{h}_r) dr
\]

and then

\[
\int_s^t (\bar{h}_r, dY_r) = (\bar{Y}_t, h_t) - (\bar{Y}_s, h_s) - \int_s^t (\bar{Y}_r, dh_r) + \int_s^t (\bar{Y}_r, \bar{h}_r) dr. \tag{3.31}
\]

Finally, the Girsanov exponential (3.28) is now rewritten using (3.31) as

\[
\Lambda_t^s = \exp \left( (\bar{Y}_t, h_t) - (\bar{Y}_s, h_s) \right) \times \exp \left( - \int_s^t (\bar{Y}_r, dh_r) + \int_s^t (\bar{Y}_r, \bar{h}_r) dr - \frac{1}{2} \int_s^t (\bar{h}_r, h_r) dr \right). \tag{3.32}
\]

**Remark 3.4.7.** The Girsanov exponential (3.31) is a version of the r.v. \( \Theta(\cdot) \) defined in the exponential equation (3.24).
3.4.3 Change of Probability Measure

To conclude the development of the robust approach, similarly to the classical nonlinear filtering approach, it would interesting to represent the solution using PDEs.

**Definition 3.4.8.** Let $h : \mathbb{R}^{+} \times \mathbb{R}^{D} \rightarrow \mathbb{R}$ have continuous partial derivatives $\frac{\partial h}{\partial r}, \frac{\partial h}{\partial x_i}, \frac{\partial^2 h}{\partial x_i \partial x_j}$ for $m = 1, 2, \ldots, M$ and define a scalar process $\{B^m_t, 0 \leq t \leq T\}$ by

$$B^m_t = h^m(t, x) = h(t, X^1_t, X^2_t, \ldots, X^D_t), \ w.p.1,$$

where $X_t$ satisfies Ito SDE (3.26). Then the stochastic differential for $B^m_t$ is given by

$$dB^m_r = \left( \frac{\partial h^m_r}{\partial r} + \sum_{k=1}^{D} f^m_r \frac{\partial h^m_r}{\partial x_k} + \frac{1}{2} \sum_{j=1}^{D} \sum_{i,k=1}^{D} \sigma^i_r \sigma^k_r \frac{\partial^2 h^m_r}{\partial x_i \partial x_k} \right) dr + \sum_{i,j=1}^{D} \sigma^i_r \frac{\partial h^m_r}{\partial x_i} dW^i_r, \quad (3.33)$$

or, in vector-matrix notation, by

$$dB^m_r = \left( \frac{\partial h^m_r}{\partial r} + \sum f^T_r \nabla h^m_r + \frac{1}{2} tr \left( g_r \nabla [\nabla h^m_r] \right) \right) dr + (\nabla h^m_r)^T \sigma_r dW_r$$

where $\nabla$ is the gradient operator, $^T$ the vector or matrix transpose operation and $tr$ the trace of the inscribed matrix, that is, the sum of its diagonal components, and

$$g_r = \sigma_r \sigma^T_r. \quad (3.34)$$

Equation (3.33) can be rewritten as

$$dB^m_r = \left( \frac{\partial h^m_r}{\partial r} + A_r h^m_r \right) dr + \sum_{i,j=1}^{D} \sigma^i_r \frac{\partial h^m_r}{\partial x_i} dW^i_r \quad (3.35)$$

$$= \left( \frac{\partial h^m_r}{\partial r} + A_r h^m_r \right) dr + (\nabla h^m_r)^T \sigma_r dW_r$$
with
\[
\mathcal{A}_r h_r^m = \sum_{k=1}^{D} f_r^m \frac{\partial h_r^m}{\partial x_k} + \frac{1}{2} \sum_{j=1}^{D} \sum_{i,k=1}^{D} \sigma_{i,j} \sigma_{r,k,j} \frac{\partial^2 h_r^m}{\partial x_i \partial x_k}
\]
\[
= f^T_r \nabla h_r^m + \frac{1}{2} tr (g_r [\nabla h_r^m])
\]

By redefining
\[
\bar{\mathcal{A}}_r h_r^m = \left( \frac{\partial h_r^m}{\partial r} + \mathcal{A}_r h_r^m \right)
\]
and
\[
\bar{Y}_r^m (\nabla h_r^m)^T \sigma_r dW_r = \left( \nabla (\bar{Y}_r^m h_r^m) \right)^T \sigma_r dW_r,
\]
then
\[
(\bar{Y}_r, dh_r) = \sum_{m=1}^{M} \bar{Y}_r^m \bar{\mathcal{A}}_r h_r^m dr + \sum_{m=1}^{M} \left( \nabla (\bar{Y}_r^m h_r^m) \right)^T \sigma_r dW_r
\]
\[
= (\bar{Y}_r, \bar{\mathcal{A}}_r h_r) dr + \left( \nabla (\bar{Y}_r, h_r) \right)^T \sigma_r dW_r.
\]

Definition 3.4.9.
\[
exp \left( (\bar{Y}_t, h_t) \right) \Lambda_t^s \triangleq exp \left( (\bar{Y}_t, h_t) \right) \Lambda_t^s \exp \left( \int_s^t c(r, X_r, Y_r) dr \right)
\]
\[
\Lambda_t^s = exp \left( -\frac{1}{2} \int_s^t (g_r \nabla (\bar{Y}_r, h_r))^T \nabla (\bar{Y}_r, h_r) dr - \int_s^t \left( \nabla (\bar{Y}_r, h_r) \right)^T \sigma_r dW_r \right)
\]
and
\[
c(r, X_r, Y_r) = \frac{1}{2} (g_r \nabla (\bar{Y}_r, h_r))^T \nabla (\bar{Y}_r, h_r) + (\bar{Y}_r, h_r) - (\bar{Y}_r, \bar{\mathcal{A}}_r h_r) - \frac{1}{2} (\bar{h}_r, h_r).
\]
The idea now is to fix a $Y$-trajectory and then, based on this trajectory, define a new law for the signal process $X$. That is why this theory is called pathwise, i.e. the obtained solution is a collection of equations, one for each fixed trajectory of $Y$ with their respective initial conditions. Then the change of measure is given by

$$
\frac{\partial \tilde{P}_{xx}}{\partial P_{xx}} \bigg|_{H_t^s} = \tilde{\Lambda}_t^s
$$

(3.40)

where $H_t^s = \sigma\{X_r, s \leq r \leq t\}$.

**Proposition 3.4.10.** Let

$$
\Delta \quad u(s, x) \triangleq \exp \left( [Y_s, h_t] \right) q(s, x)
$$

be a transformation variable with well defined conditions. Due to the change of measure (3.40) and the independence of the processes $X$ and $Y$, the transformation variable $u(s, x)$ is given by (PARDOUX, 1982)

$$
u(s, x) = \mathbb{E}_{xx} \left[ \exp \left( [Y_t, h_t] \right) \varphi(X_t) \Lambda_t^s \right]$$

(3.41)

$$
u(s, x) = \tilde{\mathbb{E}}_{xx} \left\{ \exp \left( [Y_t, h_t] \right) \varphi(X_t) \exp \left[ \int_s^t c(r, X_r, Y_r) dr \right] \right\},$$

(3.42)

where the integrand $c$ is defined by equation (3.39).

It is trivial to recognize here the generalization of the Feynman-Kac formula, i.e $u(s, x)$ must be solution of the following backward PDE:

$$
\begin{cases}
\frac{\partial u}{\partial s}(s, x) + \mathcal{A}_s^Y u(s, x) + c_s(Y_s)u(s, x) = 0, \quad s \leq t, \\
u(t, x) = \varphi(x)\exp\{(Y_t, h_t)\}, \quad s = t,
\end{cases}
$$

(3.43)

where the operator $\mathcal{A}_s^Y$ depends on the observation $Y_t$ and is defined by

$$
\mathcal{A}_s^Y(Y_s)u(s, x) = \mathcal{A}_s u(s, x) - (g_s \nabla \left( \bar{Y}_s, h_s \right))^T \nabla u(s, x).
$$

(3.44)
Let \( \varphi \in \mathcal{C}_0(\mathbb{R}^D) \) and using the results presented in [Bensoussan; Lions, 1978], to each trajectory of \( Y \), there is a unique solution of the backward equation:

\[
u \in L^2([0, t], H^1(\mathbb{R}^D)) \cap \mathcal{C}([0, t]; L^2(\mathbb{R}^D)).\]

**Theorem 3.4.11.** For all \( s \in [0, t] \), the following equality holds on the space \( W_{s0} \) – a.s.:

\[
u(s, x) = \exp \left( (Y_s, h_s) \right) \mathbb{E}_{s \mid x} \left[ \varphi(X_t) \Lambda^s_t \right]. \tag{3.45}\]

The robust representation deduction shows that by making the transformation (3.41), it is possible to reduce SPDE (3.17) to an ordinary PDE in which the observation path \( Y \in \mathcal{C}([0, \infty], \mathbb{R}^D) \) appears as a parameter. This is important for the practical viewpoint.

Observe that in the exponential term of equation (3.41) the term involving the stochastic differential \( dY_t \) disappears and, assuming sufficient smoothness for \( h \), equation (3.43) reduces to a second order parabolic deterministic PDE in which the observation path \( Y \) appears as a parameter.

[Pardoux (1982)] showed, assuming among other things that \( h, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial x}, \frac{\partial^2 h}{\partial x \partial j} \) are bounded, that the backward PDE (3.43) admits a unique solution \( \nu \) for each \( Y \) and that the function \( q \) in the forward SPDE (3.17) is the unnormalized conditional density.

In [Baras, Blankenship and Hopkins (1983)] it is considered the problem of robust filtering in the unbounded case. It shows that under certain conditions, PDE (3.43) admits a unique solution \( \nu \), but does not identify \( q \) defined by SPDE (3.17) as the unnormalized conditional density. Thus, from the point of filtering theory, their results are incomplete.

The results were improved only by [Kallianpur and Karandikar (1984)] where it was established the existence conditions of the unnormalized conditional probability density without any growing restriction of \( h(x) \). More results were obtained with the redefinition of the Brownian motion in the book of [Kallianpur and Karandikar (1988)].
3.4.4 Feynman-Kac Representation

KS formula (3.11) now must be rewritten for the new measure $\mathbb{P}'_{s_2}$ defined in equation (3.40). The solution of the nonlinear filtering problem can be rewritten in terms of Feynman-Kac formula (DEL MORAL, 2004)

$$\pi_t(\varphi) = \frac{\int_{\mathbb{R}^D} \varphi(z') e^{|(y_t, h_s)|} e^{\mathcal{U}(t, z', y_t)} \mathbb{P}'(dz')}{\int_{\mathbb{R}^D} e^{|(y_t, h_s)|} e^{\mathcal{U}(t, z', y_t)} \mathbb{P}'(dz')}$$  \hspace{1cm} (3.46)

where the new potential is defined as

$$\mathcal{U}(t, z_t, y_t) \triangleq \int_0^t c(s, z_s, y_s) \, ds.$$  \hspace{1cm} (3.47)

Therefore, the nonlinear filtering problem can be solved if it is possible to compute the new measure $\mathbb{P}'(dz') = p(t, z') \, dz'$, where $p(t, z)$ is solution of Feynman-Kac formula

$$\frac{\partial p^Y}{\partial s}(s, x) = (A^Y)^* p^Y(s, x) + c^Y p^Y(s, x), \quad p(0, x) \sim \pi_0,$$  \hspace{1cm} (3.48)

which means that now an alternative signal process $Z_t$ is obtained instead of the original $X_t$.

The practical use of this powerful representation will be discussed in Chapter 4, where the nonlinear filtering problem for continuous-time models is solved using numerical methods based on the approximation schemes presented in Chapter 2.
4 MC Filters for Continuous-Time Models

4.1 Introduction

The filtering theory is largely concerned with a recursive formula for the conditional-
posteriori statistical distribution for the signal based on prior observations. The obtention
of recursive schemes to implement the solution of the filtering problem is of paramount
importance for the development of filters applied to practical situations. The technical
difficulties in representing the solution of the problem in an analytical form, except for a
small class of systems (KALMAN; BUCY [1961]—BENES [1981]), have motivated the search
for discretization schemes aiming at approximating the results using computers. Thanks
to the joint development of computer technology and algorithm improvements, the use of
filtering theory in practical applications has become a reality.

A well succeeded procedure to approximate the solution of the nonlinear filtering
problem is the use of methods based on the Kalman-Bucy filtering approach (KALMAN;
BUCY [1961]). The so-called linearized Kalman filter (GELB et al., 1974) linearizes the
nonlinear problem within small frames and then applies the traditional Kalman filter.
This method fails drastically, however, when the linearization provides significative errors
in relation to the actual mathematical model. Improvements on the filter were made by
correcting the linearization around the estimates, the so-called extended Kalman filter
(EKF) (ANDERSON; MOORE, 1979). The simplicity of the theory about the filter and
its satisfactory results make EKF the most used filter in engineering applications, as presented in the works of Maybeck (1982), Grewal and Andrews (1993) and Hemerly (1996).

Another known method to obtain an approximated solution of the nonlinear filtering problem is the discretization of the classical filtering equation. The idea is to make discrete approximations of Kushner-Stratonovich (KUSHNER, 1967a) and Zakai (ZAKAI, 1969) equations, once they are equivalent solutions and related by simple formulas. However, the overwhelming majority of the numerical schemes deal with Zakai equation because it is linear, and this of course greatly simplifies the analysis. A well established approach is to approximate the solution of Zakai equation via the splitting-up method for solving SPDE, as presented by Bensoussan, Glowinski and Rascanu (1990), Bensoussan, Glowinski and Rascanu (1992) and Le Gland (1992). In Sun and Glowinski (1993) a pathwise approximation via operator splitting was used and in Ito and Rozovskii (2000) the time integration method was improved.

Unfortunately, in spite of its popularity, Zakai equation has serious deficiencies as a computational tool for the splitting-up methods (ITO 1996). These include the following: 1-) fast dissipation of the solution as the number of time steps grows, and 2-) the effect of intermittency which manifests itself in the appearance of rare but very large peaks. On the contrary, it appears that Kushner’s equation of nonlinear filtering is not subject to the aforementioned problems.

The representation of Zakai equation solution via the use of the Girsanov exponential leads to the representation of the nonlinear filtering solution via parabolic PDEs. The seminal approach presented by Kushner (1977) approximated the obtained trajectories using a discrete-time Markov chain, whereas Di Masi and Runggaldier (1980) approximated the conditional expectation by analogous functionals of a continuous-time Markov chain. The general result is as the discretization step goes to zero, the chain will converge weakly to the diffusion and, as a consequence, the corresponding conditional expectation. Improvements on these approaches were presented by Kushner and Yin (2003), Milstein

This chapter concerns about the numerical approximation of the nonlinear filtering problem via the use of MC methods. Important results about the discretization of the classical and robust representations of the solutions of the filtering problem will be shown and rewritten in accordance with Feynman-Kac notation. Finally, numeric results are discussed and a solution to the numerical divergence problem for a specific operation mode of the robust filter is presented.

4.2 Discrete Model of Continuous Time Filtering

Let \((\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, \mathbb{P})\) be a filtered probability space on which is given a \((D + M)\)-dimensional standard Brownian motion \(\{(W_t, V_t), \mathcal{F}_{t \geq 0}\}\) independent of \(\pi_0\), a \(\mathcal{F}_t\)-measurable \(D\)-dimensional square integrable random vector, where \(D, M > 1\).

The signal process \(\{X_t, \mathcal{F}_t\}_{t \geq 0}\) is the following unobserved Markov diffusion process represented by

\[
dX_t = \overline{f}(t, X_t) \, dt + \sigma(t, X_t) \, dW_t, \quad X_0 \sim \pi_0,
\]

where \(\overline{f} : \mathbb{R}_+ \times \mathbb{R}^D \to \mathbb{R}^D\) and \(\sigma : \mathbb{R}_+ \times \mathbb{R}^D \to \mathcal{L}(\mathbb{R}^D, \mathbb{R}^D) \simeq \mathbb{R}^{D^2}\) are globally Lipschitz functions (Liptser; Shiryayev, 1977).

The observation process \(\{Y_t, \mathcal{F}_t\}_{t \geq 0}\) satisfies the following Markov process represented by

\[
dY_t = h(t, X_t) \, dt + dV_t, \quad Y_0 = 0,
\]

where \(h : \mathbb{R}_+ \times \mathbb{R}^D \to \mathbb{R}^M\) and the Wiener process \(\{V_t\}_{t \geq 0}\) has the uniformly positive definite matrix \(R(t)\), a time-dependent diagonal covariance matrix. These functions are
globally Lipschitz and, as a consequence of the linear growth bound of $h$,

$$
\mathbb{E} \left[ \int_0^T |h(s, X_s)|^2 ds \right] < \infty.
$$

Let $\mathcal{F}_t$ be the filtration generated by $Y$ up to time $t$ and $\varphi : \mathbb{R}^D \to \mathbb{R}^M$ be a bounded continuous function. The classical filtering problem can be summarized as finding the conditional distribution of $X_t$ with respect to $\mathcal{F}_t$, i.e.

$$
\pi_t(\varphi) = \int_{\mathbb{R}^D} \varphi(x) \pi_t(dx) = \mathbb{E}[\varphi(X_t) | \mathcal{F}_t]. \tag{4.3}
$$

The normalized representation $\pi_t(\varphi)$ can be decomposed, according to KS formula [KALLIANPUR; STRIEBEL, 1968], in the following ratio

$$
\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)}, \quad \mathbb{P}\text{-as}, \tag{4.4}
$$

where

$$
\rho_t(\varphi) = \tilde{\mathbb{E}} \left[ \varphi(X_t) \Lambda_t | \mathcal{F}_0 \right] \tag{4.5}
$$

is the unnormalized representation of $\pi_t(\varphi)$ and $\tilde{\mathbb{E}}$ is the conditional expectation with respect to the measure defined by the Girsanov exponential

$$
\Lambda_t = \exp \left\{ \int_0^t h^*(s, X_s) R^{-1}(s) dY_s - \frac{1}{2} \int_0^t h^*(s, X_s) R^{-1}(s) h(s, X_s) ds \right\}. \tag{4.6}
$$

[Zakai (1969) and Bhattacharya, Kallianpur and Karandikar (1995)] showed that, under certain conditions, $\rho_t(\varphi)$ can be defined as a measured-valued process solution that uniquely satisfies the following evolution equation

$$
\rho_t(\varphi) = \pi_0(\varphi) + \int_0^t \rho_s(\mathcal{A} \varphi) ds + \int_0^t \rho_s(h^*(s, X_s) \varphi) R^{-1}(s) dY_s, \quad \text{a.s. \forall t,} \tag{4.7}
$$
with \( \varphi \) being the domain of the infinitesimal generator \( \mathcal{A} \).

The numerical solution (4.5) can be obtained via an adequate time discretization which approximates solutions of the given integrals. The basic concepts related to discretization schemes are presented in Chapter 2.

Given the time discretization \( (\tau)_\Delta \) of a bounded time interval \([0, T]\), \( T > 0 \), an approximation of the conditional expectation (4.5) can be rewritten as

\[
\rho_n^\Delta(\varphi) = \hat{E} \left[ \varphi(X_n^\Delta) \Lambda_n^\Delta \mid Y_n \right],
\]

where \( \Lambda_n^\Delta \) is some approximation of equation (4.6) and

\[
Y_n = \sigma \{ Y_0 = y_0, Y_1 = y_1, \ldots, Y_n = y_n \}.
\]

The challenge here is to look for an approximation \( \rho_n^\Delta \) of \( \rho_{\tau_n} \) involving only the increments

\[
\Delta Y_n = Y_{\tau_n} - Y_{\tau_{n-1}}
\]

of the observation process. The objective is to make this approximation be satisfactory with respect with the mean square error

\[
E \left[ (\rho_n^\Delta(\varphi) - \rho_{\tau_n}(\varphi))^2 \right],
\]

and consequently with the errors of the normalized representations

\[
E \left[ (\pi_n^\Delta(\varphi) - \pi_{\tau_n}(\varphi))^2 \right],
\]
where KS formula (4.4) can be rewritten as

$$\mathbb{E} [\varphi(X_n) | Y_n] \stackrel{\Delta}{=} \pi_n^\Delta (\varphi) = \frac{\rho_n^\Delta (\varphi)}{\rho_n^\Delta (1)}. \quad (4.10)$$

### 4.2.1 Time discretization scheme

The idea is to obtain a similar model by first approximating the original model by a discrete-time measure-valued process. The treatment that follows is standard in nonlinear filtering literature and it is essentially contained in the papers of Korezlioglu and Maziotto (1983), Picard (1984), Korezlioglu (1987) and Le Gland (1989).

**Definition 4.2.1.** For the time discretization $\tau_\Delta$, the basic set of assumptions which guarantees the description of the discretized solution of the nonlinear filtering problem is given in what follows:

(i) for any $\Delta \to 0$ there exists a transition probability kernel $\mathbb{P}^\Delta$ such that

$$\sup_{t \in [0,1]} \mathbb{E} \left[ |X_{n+t} - X_n^\Delta|^2 \right] \leq K \Delta, \quad K < \infty,$$

where $\{X_n^\Delta, n = 0, \ldots, N-1\}$ is an approximating time homogeneous Markov chain with transition probability kernel $\mathbb{P}^\Delta$ and such that $X_0^\Delta = X_0$.

(ii) it is possible to simulate r.v.s according to the law $\mathbb{P}^\Delta(x, \cdot)$ for any $x \in \mathbb{R}^D$.

**Remark 4.2.2.** An example of an approximating Markov chain $\{X_n^\Delta : n = 0, \ldots, N\}$ satisfying these assumptions is given by the Euler-Maruyama scheme, the crudest of the discretization methods that can be used in the given settings. Other time discretization schemes for diffusive signals are described in the book of Kloeden and Platen (1999).

The former discrete approximating model of KS formula (4.10) is obtained by first introducing a time discretization $(\tau)_\Delta$ of the basic model. In this way, the unnormalized
representation defined in equation (4.8), given the filtration \( \mathcal{Y}_n \), can be rewritten as

\[
\rho_n^\Delta (\varphi) = \hat{E} \left\{ \varphi(X_n) \Lambda_n^\Delta \big| Y_0 = y_0, Y_1 = y_1, \ldots, Y_n = y_n \right\},
\]

(4.11)

where the Girsanov exponential (4.6) can be rewritten as

\[
\log \Lambda_n^\Delta = n \sum_{j=1}^{n} h_j^T (X_{j-1}^\Delta) \Delta Y_j - \frac{1}{2} n \sum_{j=1}^{n} |h_j(X_{j-1}^\Delta)|^2 \Delta
\]

(4.12)

with the first integral being approximated by Euler-Maruyama scheme and

\[
\bar{h}_r^m \Delta \equiv \frac{1}{\lambda_r^m} h_r^m
\]

(4.13)

being the adjusted observation function with \( \lambda_r^m \) being the root of \( m \)th diagonal element of the covariance matrix \( R(t) \).

Equation (4.12) can be rewritten in accordance with Feynman-Kac notation (DELMORAL 2004) as

\[
\Lambda_n^\Delta = \prod_{j=1}^{n} \exp \left( \bar{h}_j^T (X_{j-1}^\Delta) \Delta Y_j - \frac{1}{2} |\bar{h}_j(X_{j-1}^\Delta)|^2 \Delta \right)
\]

\[
= \prod_{j=1}^{n} g_j^\Delta(X_{j-1}^\Delta, \Delta Y_j)
\]

(4.14)

with the so-called fitness function given by

\[
g_j^\Delta(x, \Delta y) = \exp \left( \bar{h}_j^T (x) \Delta y - \frac{1}{2} |\bar{h}_j(x)|^2 \Delta \right).
\]

(4.15)

Remark 4.2.3. The choice of the approximating function \( g_j^\Delta \) in (4.15) is not unique. By approximating \( e^x \) by Taylor’s expansion around zero and assuming \( \bar{h} \) sufficiently smooth, then (for the scalar case)

\[
g_j^\Delta(x, \Delta y) \approx \left[ 1 + \left( \bar{h}(x) \Delta y - \frac{1}{2} \bar{h}^2(x) \Delta \right) + \frac{1}{2} \left( \bar{h}(x) \Delta y - \frac{1}{2} \bar{h}^2(x) \Delta \right)^2 \right],
\]
which becomes
\[
\bar{g}^\Delta_k(x, \Delta y) = \left[ 1 + \bar{h}(x)\Delta y + \frac{1}{2} |\bar{h}(x)|^2 (\Delta y^2 - \Delta) \right]
\] (4.16)

when the terms \( \Delta y \Delta \) and \((\Delta)^2\) are discarded for a sufficiently small \(\Delta\). For a bounded function \(\bar{h}\) it is possible to choose \(\Delta\) sufficiently small so that
\[
||\bar{h}|| < \sqrt{\Delta^{-1}} \quad \text{and} \quad \bar{g}^\Delta_j(x, \Delta y) > 0.
\] (4.17)

**Remark 4.2.4.** The function \(\bar{g}^\Delta_j\) is not necessarily positive. [Picard (1984)] suggested that
\[
\bar{g}^\Delta_j(x, \Delta y) = \left[ 1 + \bar{h}^\Delta(x)\Delta y + \frac{1}{2} |\bar{h}^\Delta(x)|^2 (\Delta y^2 - \Delta) \right],
\] (4.18)

where \(\bar{h}^\Delta\) is the truncated function
\[
\bar{h}^\Delta(x) = (\bar{h}(x) \wedge \Delta^{-1/2}) \lor (-\Delta^{-1/2}).
\]

Now the unnormalized representation (4.8) can be rewritten in Lebesgue integral notations as
\[
\rho^\Delta_n(\varphi) = \int \varphi(x_n) \Lambda^\Delta_{1:n} \mathbb{P}^\Delta((X_0, \ldots, X_n) \in d(x_0, \ldots, x_n))
\] (4.19)
because now \(\{(W_t, Y_t) : t \in [0, 1]\}\) is a \((\mathbb{R}^D \times \mathbb{R}^M)\)-valued standard Brownian motion and \(X_0\) is a r.v. with law \(\pi_0\) independent of \((W_t, Y_t)\).

The Markov property states that
\[
\mathbb{P}^\Delta((X_0, \ldots, X_n) \in d(x_0, \ldots, x_n)) = \mathbb{P}^\Delta(x_{n-1}, dx_n) \ldots \mathbb{P}^\Delta(x_0, dx_1) \mathbb{P}^\Delta(dx_0).
\]
Then, by applying equation (4.15) into (4.19), the unnormalized representation becomes

$$\rho_n^\Delta(\varphi) = \int \varphi(x_n^\Delta) \prod_{j=1}^n g_j^\Delta(x_{j-1}^\Delta, \Delta y_j) \prod_{j=1}^n P^\Delta(x_{j-1}, dx_j) P^\Delta(dx_0) \quad (4.20)$$

the discrete format of Feynman-Kac formula for measure-valued system [DEL MORA L 2004].

Finally, applying equation (4.20) into (4.10), the approximated solution of the nonlinear filtering problem is given by

$$\pi_n^\Delta(\varphi) = \frac{\int \varphi(x_n^\Delta) \prod_{j=1}^n g_j^\Delta(x_{j-1}^\Delta, \Delta y_j) \prod_{j=1}^n P^\Delta(x_{j-1}, dx_j) P^\Delta(dx_0)}{\int \prod_{j=1}^n g_j^\Delta(x_{j-1}^\Delta, \Delta y_j) \prod_{j=1}^n P^\Delta(x_{j-1}, dx_j) P^\Delta(dx_0)}. \quad (4.21)$$

Next theorem states about the convergence speed of the approximation [PICARD, 1984].

**Definition 4.2.5. Regularity Hypothesis:**

(i) all the moments of $X_0$ are finite;

(ii) $\sigma$, $\overline{f}$, and $\varphi$ are $K$-Lipschitz functions for some $K > 0$;

(iii) $h$ is twice continuously differentiable, $\frac{\partial h}{\partial x}$ and $\frac{\partial^2 h}{\partial x^2}$ are bounded by $K$;

(iv) $\mathbb{E}[\Lambda_1^2] < \infty$ or square-integrable.

**Theorem 4.2.6.** Assume the regularity hypothesis from Definition 4.2.5. If $\rho^\Delta$ presented by equation (4.19) contains one of $g^\Delta$ described in one of equations (4.15), (4.16) or (4.18), then, when $\Delta \to 0$ (that is, when $N$ tends to infinity),

$$\mathbb{E} [\hat{\pi}_T(\varphi) - \hat{\pi}_N^\Delta(\varphi)]^2 = O(\Delta^2).$$

**Corollary 4.2.7.** Assume the regularity hypothesis from Definition 4.2.5 and suppose that $\rho^\Delta$ is defined by equation (4.19) containing $g^\Delta$ described in one of (4.15) or (4.18).
In the case where \( g^\Delta \) is described by (4.18) suppose moreover that \( \varphi \) is bounded. Given the approximated solution of the nonlinear filtering problem (4.21), then

\[
\hat{\mathbb{E}} \left[ \pi_T(\varphi) - \pi_{T,\Delta}^N(\varphi) \right]^2 = \mathcal{O}(\Delta)
\]

**Remark 4.2.8.** Since the discretization time \((\tau)\) induces an error of order \( \Delta \), it is not worth choosing as approximation of \( X \) which induces a smaller error!

The error bound caused by the discretization of the time interval \([0, T]\) and the approximation of the signal semigroup is well understood (see for instance Proposition 5.2 in Korezlioglu and Maziotto (1983), Theorem 2 in Picard (1984), Theorem 4.2 in Korezlioglu and Runggaldier (1993) and also Theorem 4.1 in Di Masi, Pratelli and Runggaldier (1985)). More precisely, for a time discretization \((\tau)\) the following well-known theorem is developed in Korezlioglu and Runggaldier (1993).

**Theorem 4.2.9.** Let \( \varphi \) be a bounded test function on \( \mathbb{R}^D \) satisfying the Lipschitz condition

\[
|\varphi(x) - \varphi(z)| \leq k(\varphi) |x - z|.
\]

Then, for a time discretization \((\tau)\) of a bounded interval \([0, 1]\),

\[
\sup_{t \in [0, 1]} \mathbb{E} \left[ |\pi_t(\varphi) - \pi_{T,\Delta}^N(\varphi)| \right] \leq C \sqrt{\Delta} (||\varphi|| + k(\varphi)),
\]

where \( C \) is some finite constant.

The discrete time approximating model (4.21) can be regarded as the optimal filter of a suitably defined discrete time filtering problem. In most of the applications in engineering the whole path of observation process is not completely known. Instead of that the acquisition of the observation data is made at regularly spaced times. In this situation the given approximating model and the sampled observed record \( \{ \Delta Y_n; n = 1, \ldots, N \} \) give a natural framework for formulating this filtering problem and for applying MC approaches to approximate numerically the solution.
4.2.2 Discretization of the Robust Filter

Similarly to KS formula (4.4), the robust representation (Pardoux, 1981) with respect to the solution of the nonlinear filtering problem is given by

\[ \pi_t(\varphi) = \frac{\bar{p}_t(\varphi)}{\bar{p}_t(1)}, \quad \mathbb{P}\text{-a.s.}, \tag{4.22} \]

where, for a fixed path \( Y_n \), now the unnormalized representation is defined as

\[ \bar{p}_t(\varphi) = \mathbb{E} \left\{ \exp \left[ \int_{0}^{t} c(r, X_r, Y_r)dr \right] \right\} \tag{4.23} \]

for the new measure \( \mathbb{P}^Y \) and

\[ c(r, X_r, Y_r) = \frac{1}{2} \left( g_r \nabla (\bar{Y}_r, h_r) \right)^T \nabla (\bar{Y}_r, h_r) + (\bar{Y}_r, \bar{h}_r) - (\bar{Y}_r, \mathcal{A}_r h_r) - \frac{1}{2} (\bar{h}_r, h_r), \tag{4.24} \]

in accordance with the results presented in Chapter 3.

Given a time discretization \( (\tau)_\Delta \), the robust representation (4.22) can be approximated by the discrete-time measure-valued process

\[ \mathbb{E} [\varphi(X_n)|Y_n] \triangleq \pi_n^\Delta(\varphi) = \frac{\bar{p}_n^\Delta(\varphi)}{\bar{p}_n^\Delta(1)}, \tag{4.25} \]

where

\[ \bar{p}_n^\Delta(\varphi) = \mathbb{E} \left\{ \exp \left[ \int_{0}^{\Delta} c(r, X_{\Delta}^n)dr \right] \right\} \tag{4.26} \]

with the following term being integrated according to the traditional Euler scheme:

\[ \log \bar{A}_{1:n} = \sum_{j=1}^{n} c(j, X_j^\Delta, Y_j)\Delta. \tag{4.27} \]
Equation (4.27) can be rewritten in accordance with Feynman-Kac notation as

\[
\Lambda_{1:n}^{\Delta} = \prod_{j=1}^{n} \exp \left( c(j, X_{j-1}^{\Delta}, Y_{j}) \Delta \right) \\
= \prod_{j=1}^{n} \tilde{g}_{j}^{\Delta}(X_{j-1}^{\Delta}, Y_{j}) 
\]

with the so-called \textit{fitness} function given by the deterministic term

\[
\tilde{g}_{j}^{\Delta}(x, y) = \exp \left( c(j, x, y) \right). 
\]

Rewriting the unnormalized representation (4.26) in Lebesgue integral notation as

\[
\tilde{p}_{n}^{\Delta}(\varphi) = \int \varphi(z_{n}^{\Delta}) \exp \left( (y_{n}, h_{n}) \right) \Lambda_{1:n}^{\Delta} \mathbb{P}_{Y}^{\Delta}((Z_{0}, \ldots, Z_{n}) \in d(z_{0}, \ldots, z_{n})), 
\]

the discrete format of the Feynman-Kac formulae for measure-valued system is given by

\[
\tilde{p}_{n}^{\Delta}(\varphi) = \int \varphi(z_{n}^{\Delta}) \exp \left( (y_{n}, h_{n}) \right) \prod_{j=1}^{n} \tilde{g}_{j}^{\Delta}(z_{j-1}^{\Delta}, y_{j}) \prod_{j=1}^{n} \mathbb{P}_{Y}^{\Delta}(z_{j-1}, dz_{j}) \mathbb{P}_{Y}^{\Delta}(dz_{0}). 
\]

Finally, applying equation (4.31) into equation (4.25), the approximated solution of the robust representation of the nonlinear filtering problem is given by

\[
\pi_{n}^{\Delta}(\varphi) = \frac{\int \varphi(z_{n}^{\Delta}) \exp \left( (y_{n}, h_{n}) \right) \prod_{j=1}^{n} \tilde{g}_{j}^{\Delta}(z_{j-1}^{\Delta}, y_{j}) \prod_{j=1}^{n} \mathbb{P}_{Y}^{\Delta}(z_{j-1}, dz_{j}) \mathbb{P}_{Y}^{\Delta}(dz_{0})}{\int \exp \left( (y_{n}, h_{n}) \right) \prod_{j=1}^{n} \tilde{g}_{j}^{\Delta}(z_{j-1}^{\Delta}, y_{j}) \prod_{j=1}^{n} \mathbb{P}_{Y}^{\Delta}(z_{j-1}, dz_{j}) \mathbb{P}_{Y}^{\Delta}(dz_{0})}. 
\]

4.2.3 General Recursive Filter

The solution of the nonlinear filtering problem can be seen from another point of view when a Markov chain is defined using the abstract transition distribution

\[
K_{n}(x_{n-1}, dx_{n}) = \mathbb{P}(X_{n} \in dx_{n} | X_{n-1} = x_{n-1}).
\]
For a test function $\varphi(x)$, the new abstract transition distribution can be manipulated as the following

$$
\mathbb{E}[\varphi(X_n) | X_{n-1} = x] = \int K_n(x, dy) \varphi(y) = K_n(\varphi)(y),
$$

(4.33)

what implies in

$$
\mathbb{E}[\varphi(X_{n+p}) | X_n = x] = K_{n+1}K_{n+2} \ldots K_{n+p}(\varphi)(x), \quad p = 1, 2, \ldots.
$$

In this way, the abstract transition distribution $K$ can be used on measures of the form given by

$$
\eta(\varphi) = \int \eta(dy) \varphi(y) \quad \text{with} \quad (\eta K)(dy) = \int \eta(dx) K(x, dy).
$$

Assuming now that the discrete $\eta_n$ is defined as the law of the signal process $X_n$, this yields that

$$
\eta_n(\varphi) = \mathbb{E}[\varphi(X_n)] = \mathbb{E}[\mathbb{E}[\varphi(X_n) | X_{n-1}]] \\
= \int \eta_{n-1}(dx) \mathbb{E}[\varphi(X_n) | X_{n-1} = x] \\
= \int \eta_{n-1}(dx) K_n(\varphi)(x) = (\eta_{n-1}K_n)(\varphi) \\
\Rightarrow \eta_n = \eta_{n-1}K_n = \eta_0K_1K_2 \ldots K_n.
$$

(4.34)

From now on, the assertions are simple consequences of formulae (4.33) and (4.34). With $Y = y$ fixed and given the fitness function $g_n(x) = g_n(x, y)$, the solution of the nonlinear filtering problem represented by equations (4.21) and (4.32) is defined by the following general format:

$$
\mathbb{E}[\varphi(X_0, \ldots, X_n) | Y_0 = y_0, \ldots, Y_n = y_n] = \frac{\mathbb{E}\left[\varphi(x_0, \ldots, x_n) \prod_{j=0}^n g_j(X_{p})\right]}{\mathbb{E}\left[\prod_{j=0}^n g_j(X_{j})\right]}.
$$
Proposition 4.2.10 \(\text{[Del Moral (2004)]}\). Given the assumptions of formulae \((4.33)\) and \((4.34)\), the solution of the nonlinear filtering problem is

\[
\eta_n(\varphi) = \frac{\mathbb{E} \left[ \varphi(X_n) \prod_{j=0}^{n-1} g_j(X_j) \right]}{\mathbb{E} \left[ \prod_{j=0}^{n-1} g_j(X_j) \right]}
\]

and the new measure for one-step prediction is

\[
\hat{\eta}_n(\varphi) = \frac{\mathbb{E} \left[ \varphi(X_n) \prod_{j=0}^{n-1} g_j(X_j) \right]}{\mathbb{E} \left[ \prod_{j=0}^{n} g_j(X_j) \right]}
\]

Corollary 4.2.11. The results in Proposition 4.2.10 implies in the following process:

\[
\eta_n \xrightarrow{\text{Correction/Updating}} \hat{\eta}_n \xrightarrow{\text{Prediction}} \eta_{n+1} = \hat{\eta}_n K_{n+1},
\]

where the updating is given by

\[
\Psi_n(\eta_n)(dx) = \frac{1}{\eta(g_n)} g_n(x) \eta_n(dx) \text{ (nonlinear)}
\]

and the prediction is

\[
\hat{\eta}_n K_{n+1}(dx) = \int \hat{\eta}_n(dx') K_{n+1}(x', dx') \text{ (linear)}.
\]

Proof.

\[
\hat{\eta}_n(\varphi) = \frac{\mathbb{E} \left[ \varphi(X_n) \prod_{j=0}^{n} g_j(X_j) \right]}{\mathbb{E} \left[ \prod_{j=0}^{n} g_j(X_j) \right]} = \frac{\mathbb{E} \left[ \varphi(X_n) g(X_n) \prod_{j=0}^{n-1} g_j(X_j) \right]}{\mathbb{E} \left[ g(X_n) \prod_{j=0}^{n-1} g_j(X_j) \right]} \downarrow (4.35)
\]

\[
\hat{\eta}_n(\varphi) = \frac{\mathbb{E} \left[ \varphi(X_n) g(X_n) \prod_{j=0}^{n-1} g_j(X_j) \right]}{\mathbb{E} \left[ g(X_n) \prod_{j=0}^{n-1} g_j(X_j) \right]} / \mathbb{E} \left[ \prod_{j=0}^{n-1} g_j(X_j) \right] \downarrow (4.36)
\]

\[
\hat{\eta}_n(\varphi) = \frac{\eta_n(g_n\varphi)}{\eta_n(g_n)} (4.37)
\]
By Markov property
\[
\eta_n(\varphi) = \frac{\mathbb{E}\left[\varphi(X_n) \prod_{j=0}^{n-1} g_j(X_j)\right]}{\mathbb{E}\left[\prod_{j=0}^{n-1} g_j(X_j)\right]} = \frac{\mathbb{E}\left[\mathbb{E}[\varphi(X_n)|X_{n-1}] \prod_{j=0}^{n-1} g_j(X_j)\right]}{\mathbb{E}\left[\prod_{j=0}^{n-1} g_j(X_j)\right]}
\]
\[
\downarrow
\]
\[
= \frac{\mathbb{E}\left[K_n(\varphi)(X_{n-1}) \prod_{j=0}^{n-1} g_j(X_j)\right]}{\mathbb{E}\left[\prod_{j=0}^{n-1} g_j(X_j)\right]}
\]
\[
= \hat{\eta}_{n-1}K_n(\varphi)(X_{n-1})
\]
\[
\Rightarrow \eta_n = \hat{\eta}_{n-1}K_n
\]  
(4.39)  
(4.40)  
(4.41)  
(4.42)

\[\square\]

4.3 Monte Carlo Filters

Last sections presented results stating that KS-based formulas given by equations (4.19) and (4.30) can be numerically solved using a time discretization $(\tau)_\Delta$. Here the idea is to implement the discretized solution using MC method to approximate the measures $\mathbb{P}_\Delta$ and $\mathbb{P}_1^\Delta$ in equations (4.19) and (4.30), respectively.

The basic idea of the traditional Monte Carlo filter (MCF) is to construct a measured-value process whose expectation at any time is the conditional distribution of the signal process $X$. This approximated measure is composed by the set $\tilde{X}_n = (\tilde{X}_1^n, \ldots, \tilde{X}_P^n)$ containing $P$ particles which evolve independently with the same law as $X$. These particles have associated weights containing encoded information about the observation process $Y$. In other words, the solution of the nonlinear filtering problem is given by the random measure

\[
\pi_n^{\Delta,P} = \frac{1}{P} \sum_{i=1}^{P} \delta_{\pi_n^{(i)}} \xrightarrow{P \to \infty} \pi_n \text{ (in some sense),}
\]

i.e. a $P$-approximating model for sampling $i.i.d. \sim \pi_n$ with $\delta_{\pi_n^{(i)}}$ denoting the delta-Dirac
mass located at $\pi_n^{(i)}$.

The concept of random measure can be applied to the discretized solution of the filtering problem by rewriting equation (4.21) as

$$
\pi_n^\Delta(\varphi) = \int \varphi(x_n^\Delta) \mathbb{P}^\Delta,\mathbb{P}(dx_n|y_{1:n}),
$$

(4.43)

where

$$
\mathbb{P}^\Delta,\mathbb{P}(dx_n|y_{1:n}) = \sum_{i=1}^{P} \mu_n^{(i)} \delta_{x_n^{(i)}}(d(x_{0:n}))
$$

with $\mu_n^{(i)}$ being the weight corresponding to all information with respect to the particle or realization $i$.

An immediate question is to know how the discrete time $P$-particle scheme and the time discretization $(\tau)_\Delta$ match? This result is clearly explained in the next theorem described by [Crisan, Del Moral and Lyons (1999)] and it allows the use of MCF in practical applications.

**Theorem 4.3.1.** For any bounded Lipschitz test function $\varphi$ such that

$$
|\varphi(x) - \varphi(z)| \leq k(\varphi)|x - z|.
$$

Then

$$
\sup_{t\in[0,1]} \mathbb{E} \left| \pi_t(\varphi) - \pi_n^\Delta,\mathbb{P}(\varphi) \right| \leq C_1 \sqrt{\Delta} (||\varphi|| + k(\varphi)) + C_2 \sqrt{\frac{1}{P\Delta}} ||\varphi||,
$$

where $C_1$ is the finite constant appeared in Theorem 4.2.9 and $C_2 = 2\sqrt{2}e^{12||h||^2}$. In addition, if $p = q = 1$ and $a, b, f, h$ are four times continuously differentiable with bounded derivatives, then

$$
\sup_{t\in[0,1]} \mathbb{E} \left| \pi_t(\varphi) - \pi_n^\Delta,\mathbb{P}(\varphi) \right| \leq C_3 \sqrt{\Delta} + \sqrt{\frac{1}{P\Delta}},
$$
with $C_3$ being a constant.

Analogously to the procedure developed for obtaining the new filtering equation (4.43), the robust representation given by equation (4.32) can be now rewritten in the random measure notation as

$$
\pi_n^\Delta(\varphi) = \int \varphi(z_n^\Delta) \exp \{ [y_n, h_n] \} \mathbb{P}_Y^{\Delta, P} (dz_n | y_{1:n}), \quad (4.44)
$$

where

$$
\mathbb{P}_Y^{\Delta, P} (dz_n | y_{1:n}) = \sum_{i=1}^P \tilde{p}_n^{(i)} \delta_{z_n^{(i)}} (dz_{0:n})
$$

with $\tilde{p}_n^{(i)}$ being the weight corresponding to all information with respect to the particle $i$.

### 4.3.1 The Classical MCF

The success in the obtention of the filtering solution (4.43) depends on the manner in which the measure $\mathbb{P}(dx) = p(t, x') dx'$ is computed, where in this case $p(t, x)$ is solution of the so-called Fokker-Planck equation

$$
\frac{\partial p}{\partial s}(s, x) + \mathcal{A}^* p(s, x) = 0, \quad p(0, x) \sim \pi_0. \quad (4.45)
$$

A set of $P$ independent realizations from the distribution $p(t, x)$ can be obtained using the term $\mathcal{A}$, the infinitesimal generator of the process $X$, whose solution is the SDE given by

$$
dX_t^{(i)} = f(t, X_t^{(i)}) \, dt + \sigma(t, X_t^{(i)}) \, dW_t^{(i)}, \quad i = 1, \ldots, P. \quad (4.46)
$$

The signal process defined in (4.46) is numerically implemented according to a time discretization $(\tau)_\Delta$. Finally, the numerical approximation of the conditional expectation
is computed based on the joint results of equation (4.44) and MC integration, i.e.

\[ \pi^\Delta_n(\varphi) = \sum_{i=1}^{P} \varphi(x^{(i)}_n) \mu^{(i)}_{0:n}, \]

where

\[ \mu^{(i)}_n \triangleq \frac{g^{\Delta(i)}_n}{\sum_{i=1}^{P} \Lambda^{\Delta(i)}_{1:n}} \]

with \( g^{\Delta(i)}_n \) and \( \Lambda^{\Delta(i)}_{1:n} \) being defined in equations (4.15) and (4.14), respectively.

Given the time discretization \((\tau)_{\Delta}\) and the initial distribution \(\pi_0\), the basic algorithm of the so-called classical MCF is described as follows:

---

**Classical MCF**

**Initialization**

- For \(i = 1, \ldots, P\)
  - sample \(X_{0}^{\Delta(i)} \sim \pi_0\).
- Set \(j = 1\).
- While \(j \neq N\)

**Evolution**

- For \(i = 1, \ldots, P\)
  - evolve \(X_j^{\Delta(i)}\) in accordance with the model described by (4.46).

**Importance weights evaluation**

- For \(i = 1, \ldots, P\)
  - evaluate the importance weights \(\mu^{(i)}_n\) using equation (4.48).

**Conditional law computation**

- Compute the conditional law \(\pi^\Delta_n(\varphi)\) according to (4.47);
- Set \(j = j + 1\) and go to Evolution.
Without losing generality, assume now that the $D$-dimensional signal process $X$, $D \geq 1$, is indirectly obtained through the one-dimensional observation process $Y$ described by

$$dY_t = h(X_t) \, dt + dV_t, \quad Y_0 = 0,$$

where $h : \mathbb{R}^D \rightarrow \mathbb{R}$ is a bounded continuous function and the process $\{V_t\}_{t \geq 0}$ is a standard Brownian motion.

Analogously to the last approach, the obtention of the robust solution (4.44) is directly related to the numerical computation of the measure $P_Y(dz) = p_Y(t, z') \, dz'$, where in this case $p_Y(t, x)$ is solution of the following parabolic PDE

$$\begin{cases} 
\frac{\partial p_Y}{\partial s} = (A^Y_s)^* p_Y + c_s(Y_s) p_s = 0, \quad s \leq t \\
p_Y(0, x) = p_Y^0, \quad s = 0,
\end{cases} \quad (4.49)$$

where the operator $A^Y_s$ depends on the observation $Y_t$ and now is defined by

$$A^Y_s(s, Y_s) \varphi = \frac{1}{2} \sum_{i,j} g_{ij}(s, x) \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_i f_i^Y(s, x) \frac{\partial \varphi}{\partial x_i}. \quad (4.50)$$

Equation means the infinitesimal generator of the process $Z$ whose solution is the following alternative SDE given by

$$dZ_t = f^Y(t, Z_t) \, dt + \sigma(t, Z_t) dB_t, \quad (4.51)$$

where $B_t$ is a standard Brownian motion process,

$$c_Y(s, x) = \frac{1}{2} y^2(s) \sum_{i,j} g_{ij}(s, x) \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} - y(s) A_s h - \frac{1}{2} h^2(x),$$
and

\[ \tilde{T}_i^Y(s, x) = \tilde{T}_i(s, x) - y(s) \sum_j g_{ij}(s, x) \frac{\partial h}{\partial x_j}. \] (4.52)

Therefore, MC approximation of the robust filtering consists in computing the so-called realization-based measure \( \mathbb{P}_Y^{\Delta} \) presented in equation (4.44) by sampling \( P \) independent particles from the alternative process \( Z \) defined in SDE (4.51).

In other words, based on equation (4.44), the new representation of MC approximation is given by

\[ \pi_n(\varphi) = \sum_{i=1}^{P} \tilde{\varphi}_1(z_n^{(i)}) \tilde{\mu}_0^{(i)}, \] (4.53)

where

\[ \tilde{\mu}_0^{(i)} \equiv \frac{K_n^{(i)}}{\sum_{i=1}^{P} \tilde{\varphi}_1(z_k^{(i)}) K_n^{(i)}} \] (4.54)

is the normalized weight,

\[ K_n^{(i)} = \exp \left( \int_0^{\tau_n} c_Y(r, z_r^{(i)}) dr \right) \] (4.55)

is the modulation coefficient, and

\[ \tilde{\varphi}_1(r, z) = \exp \{ y_r, h(z) \} \] and \[ \tilde{\varphi}_2(r, z) = \exp \{ y_r, h(z) \} \varphi(z). \] (4.56)

As quoted by Davis (1981a), the influence of the observed sample path here is twofold: first, the value of \( y(r) \) appears explicitly in equation (4.53), and, secondly, the sample path \( \{ y(s), 0 \leq r \leq \tau_n \} \) determines the generator of \( Z_{\tau_n} \) via (4.79) and (4.52) and hence affects the distributions of \( K_{\tau_n} \).

The basic algorithm of this filter is given as follows:
Pathwise MCF

Initialization

• For $i = 1, \ldots, P$
  – sample $Z_0^{\Delta (i)} \sim \pi_0$.
• Set $j = 1$.
• While $j \neq N$

Evolution

• For $i = 1, \ldots, P$
  – evolve $Z_j^{\Delta (i)}$ in accordance with the model described by (4.51).

Importance weights evaluation

• For $i = 1, \ldots, P$
  – evaluate the importance weights $\overline{\mu}_n^{(i)}$ using equation (4.54).

Conditional law computation

• Compute the conditional law $\pi_n^{\Delta}(\varphi)$ according to (4.33);
• Set $j = j + 1$ and go to Evolution.

This approach is feasible in the sense that one carries it out and gets a return directly related to the amount of computational effort invested. However, the convergence could still be quite slow due to bad initializations of the process $Z$, as explained by O’Loghlen and Wright (1982) and Souza (1992).

4.4 Branching and Interacting Particle Filters

In the important papers of Korezlioglu and Maziotte (1983) and Le Gland (1984), and after in Korezlioglu (1987), Le Gland (1989) and Crisan, Del Moral and Lyons (1999), the authors proposed a preliminary time discretization scheme of the optimal filter evolution.
Then they solved the discrete-time approximating model by using MC simulations or spatial quantizations of the signal. In these papers special attention was given on the convergence rate of the time discretization scheme to the continuous-time original model. MC approach described therein consists of independent particles weighted by exponentials. Unfortunately MC approximation may not be efficient, mainly because the particles are independent of each other and the growth of the exponential weights is difficult to control as time goes by. In Del Moral (1996b) and Del Moral (1998) a way is proposed to regularize these exponential weights and a natural ergodic assumption was introduced on the signal semigroup under which the particle approximation converges in law to the optimal filter uniformly with respect to time, the so-called interacting particle filters (IPF). These MC approximations are the crudest of the random particle systems approaches.

It has recently been emphasized that a more efficient approach is to use interacting and branching particle systems to numerically solve the filtering problem. Crisan and Lyons (1997) and Crisan, Gaines and Lyons (1998) have constructed a branching particle system approximation for the filtering problem of diffusions, the so-called branching particle filter (BPF).

Crisan (2003) described a system of moving branching particles whose empirical distribution at time \( t \), denoted by \( U_t^{(i)} \), converges a.s. to \( \rho_t \) defined by equation (4.7), i.e.

\[
\lim_{t \to \infty} (U_t^{(i)}, \varphi) = \tilde{E} \left[ \varphi(\cdot) \right] = \rho_t(\varphi).
\] (4.57)

The particles move according to the law of the signal \( X \) independently of each other and after fixed-length intervals will branch. The mean number of offspring of a particle will depend on the inter-branching interval of its trajectory and on the observation process, whereas the variance of the branching mechanism is the minimum possible.

In fact, Crisan and Lyons (1997) early proposed a sequence \( U^{(i)} \) of similar branching particle systems where the variance of the branching mechanism was given a priori and only the (conditional) expectation of that sequence converged to \( \rho_t \) due to an extra degree
of randomness introduced. Then, a whole set of copies of the particle system was needed in order to obtain a good approximation to the solution of Zakai equation [ZAKAI 1969]. More recently, the new approach presented in Crisan, Gaines and Lyons (1998) converged directly to $\rho_t$ without any estimate of an average, but the convergence rate of the mean square error was not deduced correctly. In Crisan (2003) finally it was deduced the exact convergence rate of the mean square error and proposed variations of the branching algorithm with improvements.

### 4.4.1 The Branching Particle Filter Algorithm

Given the time discretization $(\tau)_{\Delta}$, the *inter-branching* time interval is defined by

$$\Delta T_l = \Delta t_{l+1} - \Delta t = n\delta, \ l, n \in \mathbb{N},$$

and can be visualized in Figure 4.1 below.

![Figure 4.1 - Discretization of the time line in BPF approach.](image)

Crisan and Lyons (1997) have constructed a measured value process whose expectation at any time is the conditional distribution of $X_t$. This is a branching particle system approximation and the particles evolve independently moving with the same law as $X$ and branch according to a mechanism that depends on the trajectory of the particle and $Y$, but is independent of the events elsewhere in the system. The mechanism is chosen so that it has finite second moment and the mean number of offspring for a particle given the $\sigma$-field $\mathcal{F}_{\Delta t_{l+1}} = \sigma(\mathcal{F}_s, s < \Delta t_{l+1})$ of events up to time $\Delta t_{l+1}$ is

$$\mu_{\Delta t_{l}}^{(i)} = \exp \left( \int_{\Delta t_l}^{\Delta t_{l+1}} h^T(X_t^{(i)})dY_t - \frac{1}{2} \int_{\Delta t_l}^{\Delta t_{l+1}} \| h(X_t^{(i)}) \|^2 dt \right).$$
The variance $\nu^{(i)}$ must be minimal and consistent with the number of offspring being an integer, i.e.

$$\nu^{(i)}_{\Delta T_i} = (\mu^{(i)}_{\Delta T_i} - [\mu^{(i)}_{\Delta T_i}]) ([\mu^{(i)}_{\Delta T_i}] + 1 - \mu^{(i)}_{\Delta T_i}) \leq \frac{1}{4},$$

where $[\mu^{(i)}_{\Delta T_i}]$ is the largest integer smaller than $\mu^{(i)}_{\Delta T_i}$. The result is a cloud of paths where those surviving to the current time provide an estimate for the conditional distribution of $X_t$ in the following manner

$$U_t^{P(\Delta_i)} = \frac{1}{P(\Delta_i)} \sum_{i=1}^{P(\Delta_i)} \mu^{(i)}_{\Delta_i: \Delta_{i+1}} \delta_{X^{(i)}(\Delta_{i+1})}; \quad (4.59)$$

where $P(\Delta_i)$ is the number of particles corresponding to the instant $t = \Delta_i$.

**Corollary 4.4.1 (Crisan (2003)).** If the length of the inter-branching times is $\frac{1}{P^{\alpha}}$, where $\alpha \in \left(\frac{2}{3}, 2\right)$, then

$$\lim_{P \to \infty} P^{1-\frac{3}{2\alpha}} E[(U_t^{P}, \varphi) - \rho_t(\varphi)]^2 = c_U(t), \quad (4.60)$$

where $c_U(t)$ is a constant independent of $P$.

According to Corollary 4.4.1, the larger the length of the inter-branching times $\Delta T$ is, the better the convergence rate is. However, the order of the length of the inter-branching times cannot be larger than $\frac{1}{P^{\alpha}}$ as the last part of the system evolution is not corrected and hence a bias is introduced. By other hand, if the inter-branching times are of order $\frac{1}{P^{1/2}}$, then $U^P$ no longer converges to $\rho_t$. By branching so often, the randomness introduced in the system at branching times overpowers the corrective effect and, as a result, just as in the case when the branching variance is fixed, the limiting process is a measure valued process whose conditional expectation given the environment $Y$ is $\rho_t$ (CRISAN, 2003).

Let $\{U_t^{(i)}, \mathcal{F}_t; 0 \leq t \leq 1\}$ be a sequence of branching particle systems defined according to (4.59) and let $P(T_0)$ be the initial number of particles. The step-by-step algorithm is described as follows:
Initialization

- For $i = 1, \ldots, P$
  - sample $X_0^{(i)} \sim \pi_0$;
- Compute $\Delta \tau_0$ according to a priori convergence rate and $P(\Delta_0)$.
- Set $t = \Delta_1$.
- While $t \leq \Delta_M$, $\Delta_M = \tau_N$,

Evolution

- For $i = 1, \ldots, P(\Delta_{l-1})$
  - evolve $X_{\Delta \tau_{l-1}}^{(i)}$ in accordance with the model described by (4.46).

Importance weights evaluation

- For $i = 1, \ldots, P(\Delta_{l-1})$
  - evaluate the importance weights $\mu_{\Delta \tau_{l-1}}^{(i)}$ using equation (4.48).

Conditional law computation

- Compute the conditional law $U^{P(\Delta_l)}$ according to (4.51).

Offspring computation

- For $i = 1, \ldots, P(\Delta_{l-1})$
  - sample $\epsilon \sim U[0, 1]$;
  - compute $m^{(i)}(\Delta_l) = \text{round}(\mu_{\Delta \tau_l}^{(i)} + \epsilon - 0.5)$, with $m^{(i)}(\Delta_l)$ particles.

Replacement

- For $i = 1, \ldots, P(\Delta_{l-1})$
  - replace the particles according to $m^{(i)}(\Delta_l)$.

Interbranching interval computation

- Compute the total number of particles as $P(\Delta_l) = \sum_{i=0}^{P(\Delta_{l-1})} m^{(i)}(\Delta_l)$;
- Compute $\Delta \tau_l$ according to a priori convergence rate and $P(\Delta_l)$;
- Set $t = \Delta_{l+1}$ and go to Evolution.

Specifically, in order to keep the number of particles limited, a range containing the
minimum and the maximum number of particles for BPF algorithm must be established. According to preliminary tests, in this approach a process can be completely killed or the number of particles can be large enough to difficult applications in real-time problems. When the number of particles at time $\Delta t$ is larger than an upper limit, then the particles are not replaced for new ones; when the number of particles at time $\Delta t$ is smaller than a downer limit, then the particles are replaced for the ones with higher likelihood (Jacob; Yoneyama, 2004).

4.4.1.1 Modified Branching Particle Filter Algorithm

The idea here is to maintain the number of particles fixed during the evolution of the branching system, i.e. $P(\Delta t) = P$ for all $t$. For this case, it is not necessary to compute a new number of offspring and the replacement procedure must be modified. Based on the resampling ideas presented by (Doucet, Freitas and Gordon, 2001b), the new step-by-step algorithm for the so-called Modified Branching Particle Filter (MBPF) is presented as follows:

---

Modified BPF

 Initialization

- For $i = 1, ..., P$
  - sample $X_{0}^{(i)} \sim \pi_0$.  
- Compute the fixed $\Delta r$ according to a priori convergence rate and $P$.  
- Set $t = \Delta_1$.
- While $t \leq \Delta M$, $\Delta M = \tau_N$,  

 Evolution

- For $i = 1, ..., P(\Delta_{l-1})$
  - evolve $X_{\Delta_{l-1}}^{(i)}$ in accordance with the model described by (4.46).  

 Importance weights evaluation

- For $i = 1, ..., P(\Delta_{l-1})$
  - evaluate the importance weights $\mu_{\Delta_{l-1}}^{(i)}$ using equation (4.48).
Conditional law computation

• Compute the conditional $U_{\Delta \tau_i}^{P(\Delta_i)}$ law according to (4.39).

Weights normalization

• Compute the normalization factor $\gamma = \sum_{i=1}^{P} \mu_{\Delta \tau_{i-1}}^{(i)}$;
• For $i = 1, \ldots, P$
  – normalize the weights $\tilde{\mu}_{\Delta \tau_{i-1}}^{i} = \gamma^{-1} \mu_{\Delta \tau_{i-1}}^{(i)}$.

Replacement

• For $i = 1, \ldots, P(\Delta_{i-1})$
  – replace proportionally the particles using the normalized weights;
• Set $t = \Delta_{t+1}$ and go to Evolution.

4.4.2 General Recursive Filter

The idea is to obtain a $\mathbb{R}^D$-valued Markov model $\tilde{X}_n = (\tilde{X}_n^1, \ldots, \tilde{X}_n^P)$ such that

$$\eta_n^P = \frac{1}{P} \sum_{i=1}^{P} \delta_{\eta_n^i} \xrightarrow{P \to \infty} \eta_n \text{ (in some sense)}$$

i.e. a $P$-approximating model for sampling i.i.d. $\sim \eta_n$.

The particles are described as

$$\tilde{X}_n = (\tilde{X}_n^1, \ldots, \tilde{X}_n^i, \ldots, \tilde{X}_n^P) \xleftarrow{P \text{ particles}} \text{configuration/population/system},$$

where $i$ is the particle label, $n$ is the time parameter, and $P$ is the size of the system or precision parameter.

The general recursive filter is given by the following recursion:
Initially \((n = 0)\): \(\tilde{X}_0 = (\tilde{X}_0^1, \ldots, \tilde{X}_0^P)\) i.i.d. \(\sim \eta_0\)

Law of Large Numbers \(\Rightarrow\) \(\eta_0^p = \frac{1}{P} \sum_{i=1}^{P} \delta_{\tilde{X}_i^0} \xrightarrow{P \to \infty} \eta_0\)

Step \(n \to (n + 1)\):

\[\eta_n^p = \frac{1}{P} \sum_{i=1}^{P} \delta_{\tilde{X}_i^n} \rightarrow \eta_n \xrightarrow{\text{cont.}} \Phi_{n+1}(\eta_n) = \eta_{n+1}\]

\(\downarrow\) (given \(\tilde{X}_n\))

\(\tilde{X}_{n+1} = (\tilde{X}_{n+1}^1, \ldots, \tilde{X}_{n+1}^P)\) i.i.d. \(\sim \Phi_{n+1} \left( \frac{1}{P} \sum_{i=1}^{P} \delta_{\tilde{X}_i^n} \right)\)

\[\eta_{n+1}^p = \frac{1}{P} \sum_{i=1}^{P} \delta_{\tilde{X}_{n+1}^i} \xrightarrow{P \to \infty} \eta_{n+1}\]

which implies in

\[\mathbb{P}(\tilde{X}_0 \in d(x^1, \ldots, x^P)) = \prod_{p=1}^{n} \eta_0(dx^p)\]

and

\[\mathbb{P}(\tilde{X}_n \in d(x^1, \ldots, x^P) | \tilde{X}_{n-1} \in d(z^1, \ldots, z^P)) = \prod_{p=1}^{n} \Phi_n \left( \frac{1}{P} \sum_{i=1}^{P} \delta_{\tilde{X}_i^n} \right) (dx^p).\]

Then it is possible to conclude that

- the initial configuration \(\tilde{X}_0 = (\tilde{X}_0^1, \ldots, \tilde{X}_0^P)\) consists of \(P\) i.i.d. with law \(\eta_0\).
- given the empirical measure \(\eta_{n-1}^p = \frac{1}{P} \sum_{i=1}^{P} \delta_{\tilde{X}_{i-1}^n}\), the next configuration \(\tilde{X}_n = (\tilde{X}_{n-1}^1, \ldots, \tilde{X}_{n-1}^P)\) consists of \(P\) i.i.d. with law \(\Phi(\eta_{n-1}^p)\).
In other words, the approximating $P$-particle model can be described as (Del Moral, 2004):

\[
\begin{align*}
\eta_0 & \quad \Phi_1(\eta_0^P) & \quad \Phi_2(\eta_1^P) & \quad \Phi_n(\eta_{n-1}^P) \\
\downarrow \text{(P-i.i.d.)} & \quad \downarrow \text{(P-i.i.d.)} & \quad \downarrow \text{(P-i.i.d.)} & \quad \downarrow \text{(P-i.i.d.)} \\
\eta_0^P & \quad \eta_1^P & \quad \eta_2^P & \quad \eta_n
\end{align*}
\]

4.5 Numerical Results

This section aims at comparing the estimate performances of MC methods with that obtained by the traditional EKF. Generally, for the nonlinear filters, just parameters related to the initial number of particles and length of the inter-branching time can be adjusted for improving the convergence rate, except for MCF where just the number of particles is important. The initial covariance matrix of EKF must be adjusted with accuracy in order to minimize the transient in the first simulation steps. The criteria used for the comparison and the numerical examples are presented next.

At a fixed time instant $\tau$, the methodology computes the mean absolute or the square error criteria between the first moment estimate of the signal, $\hat{x}_\tau$, and the correspondent real signal, $x_\tau$. For MC filters the performance is also evaluated for different numbers of particles.

4.5.1 Confidence Intervals

Given the $j$th sample at time $\tau$ from the pdf of the process $X$, the $j$th estimate of the generalized error is defined as

\[
\hat{\varepsilon}^{(j)}_\tau = \frac{1}{M} \sum_{k=1}^{M} \psi(\hat{x}^{(j,k)}_\tau - x^{(j,k)}_\tau), \quad k = 1, ..., M,
\]

where $\psi : \mathbb{R} \to \mathbb{R}$, and $M$ is the total number of different simulations of sample paths of the real signal $x^{(j,k)}_\tau$ and the estimated signal $\hat{x}^{(j,k)}_\tau$. The error $\hat{\varepsilon}_j$ is a r.v. independent
and Gaussian for large $M$.

In order to construct a confidence interval for the generalized error \( 4.61 \), $R$, $j = 1, \ldots, R$, different initial conditions for $X_0 \sim \pi_0$, and the respective real signals and estimates are simulated. The mean of the batch averages is

$$\hat{\varepsilon}_\tau = \frac{1}{R} \sum_{j=1}^{R} \hat{\varepsilon}^{(j)}_\tau = \frac{1}{RM} \sum_{j=1}^{R} \sum_{k=1}^{M} |\hat{x}^{(j,k)}_\tau - \hat{x}^{(j,k)}_\tau|$$

(4.62)

and the formula

$$\hat{\sigma}^2_\varepsilon \triangleq \frac{1}{R-1} \sum_{j=1}^{R} (\hat{\varepsilon}^{(j)}_\tau - \hat{\varepsilon}_\tau)^2$$

(4.63)

is the estimated variance of $\hat{\varepsilon}_\tau$.

Then, the Student t-distribution with $R - 1$ degrees of freedom is used to construct the significance level $(1 - \alpha)\%$ for $\hat{\varepsilon}$ (BOX; HUNTER; HUNTER, 1978), where the confidence interval is defined as $(\hat{\varepsilon} - \Delta \hat{\varepsilon}, \hat{\varepsilon} - \Delta \hat{\varepsilon})$ with

$$\Delta \hat{\varepsilon} = t_{1 - \alpha/2, R - 1} \frac{\hat{\sigma}_\varepsilon}{\sqrt{R}}.$$  

(4.64)

In this way, the total length of the confidence interval around $\hat{\varepsilon}$ is given by

$$[\Delta \hat{\varepsilon}] = 2 \Delta \hat{\varepsilon}.$$  

(4.65)

**4.5.1.1 Performance Criteria**

By using the same procedure as presented in calculating the confidence interval for the generalized estimated error, the *absolute error criterion* defines the generalized function $\psi$ as

$$\psi(x) = |x|,$$

(4.66)
i.e. the absolute value of \( x \), whereas the *squared error criterion* is given by

\[
\psi(x) = (x)^2. \tag{4.67}
\]

The choice of the criterion depends on what the performance test intends to obtain. The absolute error suggests that the *outliers* may not have influence on the performance evaluation, instead of using the squared error.

### 4.5.2 Performance Study of the Classical Filter

Here the objective is to apply the absolute error criterion in order to compare the performance among MCF, BPF and MBPF, assuming the traditional EKF as a benchmark.

#### 4.5.2.1 Stable System

Consider the filtering problem with the following one-dimensional Ornstein-Uhlenbeck process

\[
dX_t = -1.0 \, X_t \, dt + 0.25 \, dW_t \tag{4.68}
\]

whose one-dimensional observation process is

\[
dY_t = h(X_t) \, dt + dV_t \tag{4.69}
\]

with

\[
h(x) = \begin{cases} 
\sin(x) & \text{if } |x| \leq \pi/2 \\
+1 & \text{if } x > \pi/2 \\
-1 & \text{if } x < -\pi/2 
\end{cases}
\tag{4.70}
\]
where $W_t$ and $V_t$ are independent one-dimensional standard Brownian motions. The filtering was carried out for the range $t \in [0, 5]$ which was divided in $N = 2^{11}$ intervals. All the integrals were approximated by the traditional Euler scheme. The signal process initialization was set to $X_0 \sim \mathcal{N}(2.0, 0.20)$ in order to evaluate the performance of the Euler-Maruyama discretization scheme over the nonlinear range of the observation process.

BPF and MBPF implementation were made aiming at obtaining the best convergence rate in accordance with equation (4.60) at Corollary 4.4.1. However, to simplify the algorithm implementation, the inter-branching times computed were multiple of $2^k$, $k = 0, 1, 2, \ldots$.

Figure 4.2 presents a typical realization of the signal process (4.68) with the filtering results for MCF with 100 particles, BPF initialized with 100 particles and range from 1 up to 1000, and EKF.

![Graph of signal process with filtering results](image)

**FIGURE 4.2** – A realization of (4.68) and its respective MCF, BPF and EKF estimates.

The performance evaluation was made based on the robustness of MCF, MBPF and BPF estimates, where the formers were tested for $P = 10, 100, 1000$, and the latter with a given initial number of particles inside a range of working. EKF robustness estimate was evaluated for 100 runs. To satisfy the absolute error criterion, the absolute error $\varepsilon$ and its respective 90% confidence interval $[\Delta \varepsilon]$ were computed at instants $t = 1.0s$ and $t = 3.0s$, as presented in Tables 4.1 and 4.2 respectively. The performance provided by
the nonlinear filters and EKF are similar, except by the fact that here MCF, MBPF and
BPF confidence interval at \( t = 1.0s \) are significantly better for \( P \geq 100 \), once the signal
process is located at the strong nonlinear region of the observation one, i.e. MC-based
filters seem to have better performance in regions where the observation is close to the
saturation when compared to EKF.

TABLE 4.1 – Estimation and 90% confidence interval of \( \hat{\varepsilon} \) obtained by MCF, BPF, MBPF
and EKF at \( t = 1.0s \) for the stable system.

<table>
<thead>
<tr>
<th>P</th>
<th>( \hat{\varepsilon}(10^{-1}) )</th>
<th><a href="10%5E%7B-1%7D">( \Delta \hat{\varepsilon} )</a></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>MCF</td>
</tr>
<tr>
<td>10</td>
<td>2.155±0.230</td>
<td>(2.117,2.193)</td>
</tr>
<tr>
<td>100</td>
<td>2.043±0.151</td>
<td>(2.018,2.068)</td>
</tr>
<tr>
<td>1000</td>
<td>2.023±0.141</td>
<td>(2.000,2.046)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MBPF</td>
</tr>
<tr>
<td>10</td>
<td>2.794±0.458</td>
<td>(2.718,2.870)</td>
</tr>
<tr>
<td>100</td>
<td>2.108±0.156</td>
<td>(2.082,2.134)</td>
</tr>
<tr>
<td>1000</td>
<td>2.044±0.140</td>
<td>(2.021,2.067)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BPF</td>
</tr>
<tr>
<td>10 [5 15]</td>
<td>2.469±0.316</td>
<td>(2.417,2.521)</td>
</tr>
<tr>
<td>10 [1 1000]</td>
<td>2.302±0.281</td>
<td>(2.255,2.349)</td>
</tr>
<tr>
<td>100 [50 200]</td>
<td>2.255±0.189</td>
<td>(2.220,2.229)</td>
</tr>
<tr>
<td>100 [1 1000]</td>
<td>2.075±0.158</td>
<td>(2.049,2.101)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>EKF</td>
</tr>
<tr>
<td></td>
<td>2.252±0.158</td>
<td>(2.226,2.278)</td>
</tr>
</tbody>
</table>

TABLE 4.2 – Estimation and 90% confidence interval of \( \hat{\varepsilon} \) obtained by MCF, BPF, MBPF
and EKF at \( t = 3.0s \) for the stable system.

<table>
<thead>
<tr>
<th>P</th>
<th>( \hat{\varepsilon}(10^{-1}) )</th>
<th><a href="10%5E%7B-1%7D">( \Delta \hat{\varepsilon} )</a></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>MCF</td>
</tr>
<tr>
<td>10</td>
<td>1.491±0.114</td>
<td>(1.472,1.510)</td>
</tr>
<tr>
<td>100</td>
<td>1.417±0.102</td>
<td>(1.400,1.434)</td>
</tr>
<tr>
<td>1000</td>
<td>1.402±0.104</td>
<td>(1.385,1.419)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MBPF</td>
</tr>
<tr>
<td>10</td>
<td>1.567±0.099</td>
<td>(1.551,1.583)</td>
</tr>
<tr>
<td>100</td>
<td>1.441±0.110</td>
<td>(1.423,1.459)</td>
</tr>
<tr>
<td>1000</td>
<td>1.417±0.109</td>
<td>(1.399,1.435)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>BPF</td>
</tr>
<tr>
<td>10 [5 15]</td>
<td>1.563±0.122</td>
<td>(1.543,1.583)</td>
</tr>
<tr>
<td>10 [1 1000]</td>
<td>1.495±0.115</td>
<td>(1.476,1.514)</td>
</tr>
<tr>
<td>100 [50 200]</td>
<td>1.477±0.113</td>
<td>(1.458,1.496)</td>
</tr>
<tr>
<td>100 [1 1000]</td>
<td>1.420±0.106</td>
<td>(1.403,1.437)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>EKF</td>
</tr>
<tr>
<td></td>
<td>1.419±0.104</td>
<td>(1.402,1.436)</td>
</tr>
</tbody>
</table>
CHAPTER 4. MC FILTERS FOR CONTINUOUS-TIME MODELS

Note that MCF obtained better results when compared to the other nonlinear filter for the same number of initial particles. However, the convergence rate of BPFs can be better than those presented by MC-based ones for an adequate choice of the inter-branching times and number of particles. The worst results of the performance evaluation for the branching particle system must be investigated better, but some explanations about the practical implementation can be given: 1-) the inter-branching interval computed was multiple of \( 2^k \), what cannot be the right choice for this kind of model, and 2-) the results show that the range limitation of the particles can make the performance of the estimates worse in cases where the number of particles should be greater than the computed one.

4.5.2.2 Unstable System

Now the one-dimensional Ornstein-Uhlenbeck process is defined as

\[
 dX_t = 0.5 X_t dt + 0.25 dW_t,
\]

where \( W_t \) is a one-dimensional standard Brownian motion and \( X_0 \sim \mathcal{N}(0.0; 0.20) \). The observation process is the one given by equations (4.69) and (4.70). Figure 4.3 shows a typical realization of the given signal process (4.71) and the estimates provided by MCF with 100 particles, BPF initialized with 100 particles and range from 1 up to 1000, and EKF.

![Figure 4.3](image)

FIGURE 4.3 – A realization of (4.71) and its respective MCF, BPF and EKF estimates.
The performance evaluation here is analogous to the last example, except that now the simulation was run for a time range where the signal process \( X \) cannot be observed through the observation process \( Y \), i.e. \( x > \pi/2 \). Differently from the stable case, EKF failed to provide an acceptable bounded error estimate when compared to the nonlinear filters at time \( t = 3.0 \)s, as presented by the absolute errors \( \hat{\varepsilon} \) in Tables 4.3 and 4.4.

| TABLE 4.3 – Estimation and 90% confidence interval of \( \hat{\varepsilon} \) obtained by MCF, MBPF, BPF and EKF at \( t = 1.0 \)s for the unstable system. |
|---|---|---|
| P \( \times 10^3 \) | \( \hat{\varepsilon} \) | \( [\Delta \varepsilon] \) |
| **MCF** | | |
| 10 | 1.576±0.224 | (1.539,1.613) |
| 100 | 1.440±0.117 | (1.421,1.460) |
| 1000 | 1.419±0.104 | (1.402,1.436) |
| **MBPF** | | |
| 10 | 2.001±0.260 | (1.917,2.085) |
| 100 | 1.562±0.130 | (1.540,1.584) |
| 1000 | 1.426±0.109 | (1.408,1.444) |
| **BPF** | | |
| 10 | 2.049±0.357 | (1.990,2.108) |
| 100 | 1.843±0.300 | (1.793,1.893) |
| 1000 | 1.738±0.175 | (1.709,1.767) |
| 1000 | 1.475±0.118 | (1.455,1.494) |
| **EKF** | 2.807±0.206 | (2.732,2.882) |

| TABLE 4.4 – Estimation and 90% confidence interval of \( \hat{\varepsilon} \) obtained by MCF, MBPF, BPF and EKF at \( t = 3.0 \)s for the unstable system. |
|---|---|---|
| P \( \times 10^3 \) | \( \hat{\varepsilon}(10^{-1}) \) | \( [\Delta \varepsilon](10^{-1}) \) |
| **MCF** | | |
| 10 | 7.027±0.825 | (6.890,7.164) |
| 100 | 6.588±0.566 | (6.494,6.682) |
| 1000 | 6.511±0.529 | (6.423,6.599) |
| **MBPF** | | |
| 10 | 7.461±0.854 | (7.319,7.603) |
| 100 | 6.902±0.608 | (6.801,7.003) |
| 1000 | 6.570±0.529 | (6.482,6.658) |
| **BPF** | | |
| 10 | 7.366±0.889 | (7.219,7.513) |
| 100 | 7.494±1.010 | (7.326,7.662) |
| 1000 | 6.623±0.547 | (6.532,6.714) |
| 1000 | 6.651±0.574 | (6.556,6.746) |
| **EKF** | 10.900±0.730 | (9.482,12.318) |
4.5.3 Performance Study of the Pathwise Filter

The main contribution here is to compare the performances of the pathwise MCF and EKF approaches in the filtering of nonlinear models. Specifically, for the nonlinear filter just the number of particles after each filter initialization can be modified, whereas the initial covariance matrix of EKF must be adjusted with accuracy in order to minimize the transient in the estimates during the first steps.

4.5.3.1 Stable System

Considering first the system suggested in [Davis (1981a)], analogously to the last section, let the filtering problem be applied to the following one-dimensional Ornstein-Uhlenbeck process

\[
dX_t = -1.0 X_t \, dt + 0.25 \, dW_t \quad (4.72)
\]

whose associated one-dimensional observation process is

\[
dY_t = h(X_t) \, dt + dV_t \quad (4.73)
\]

with

\[
h(x) = \begin{cases} 
\sin(x) & \text{if } |x| \leq \pi/2 \\
+1 & \text{if } x > \pi/2 \\
-1 & \text{if } x < -\pi/2 
\end{cases}
\quad (4.74)
\]

where \( W_t \) and \( V_t \) are independent one-dimensional standard Brownian motions. The filtering was carried out for \( t \in [0, 5] \) by using a time step of \( \Delta = 2^{-11} \) s. The signal initialization was set to \( X_0 \sim N(2.0, 0.25) \) in order to demonstrate the performance of the Euler-Maruyama discretization scheme over a small part of the saturated range of the observation process.
The performance evaluation was made based on the robustness of the pathwise MCF and EKF estimates, where the former was tested by using $P = 10, 100, 1000$. To satisfy the absolute and squared error criteria, the respective estimated errors $\hat{\varepsilon}^a$ and $\hat{\varepsilon}^s$ were computed for 90% confidence interval $[\Delta \hat{\varepsilon}^a]$ and $[\Delta \hat{\varepsilon}^s]$ at two different time instants: $t = 1.0s$, where the idea is to evaluate the filter performance over the nonlinear range of the observation process, and $t = 3.0s$, the approximated linear range. Tables 4.5 and 4.6 present, respectively, the final results, where each configuration was tested for $M = 100$ and $R = 100$ realizations, i.e. $M$ different realizations for each one of the $R$ different initializations of the filters.

**TABLE 4.5** – Estimation and 90% confidence interval of $\hat{\varepsilon}^a$ (x10$^{-1}$) and $\hat{\varepsilon}^s$ (x10$^{-2}$) obtained by MCF and EKF at $t = 1.0s$ for the stable system.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\hat{\varepsilon}^a$</th>
<th>$[\Delta \varepsilon^a]$</th>
<th>$\hat{\varepsilon}^s$</th>
<th>$[\Delta \varepsilon^s]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>2.138±0.227</td>
<td>0.325</td>
<td>7.244±1.450</td>
<td>2.074</td>
</tr>
<tr>
<td>100</td>
<td>2.032±0.152</td>
<td>0.061</td>
<td>6.483±0.923</td>
<td>0.368</td>
</tr>
<tr>
<td>1000</td>
<td>2.010±0.140</td>
<td>0.017</td>
<td>6.433±0.912</td>
<td>0.113</td>
</tr>
<tr>
<td>EKF</td>
<td>2.252±0.158</td>
<td>0.063</td>
<td>8.024±1.111</td>
<td>0.443</td>
</tr>
</tbody>
</table>

**TABLE 4.6** – Estimation and 90% confidence interval of $\hat{\varepsilon}^a$ (x10$^{-1}$) and $\hat{\varepsilon}^s$ (x10$^{-2}$) obtained by MCF and EKF at $t = 3.0s$ for the stable system.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\hat{\varepsilon}^a$</th>
<th>$[\Delta \varepsilon^a]$</th>
<th>$\hat{\varepsilon}^s$</th>
<th>$[\Delta \varepsilon^s]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.544±0.113</td>
<td>0.162</td>
<td>3.716±0.486</td>
<td>0.695</td>
</tr>
<tr>
<td>100</td>
<td>1.426±0.102</td>
<td>0.041</td>
<td>3.184±0.418</td>
<td>0.167</td>
</tr>
<tr>
<td>1000</td>
<td>1.405±0.107</td>
<td>0.013</td>
<td>3.117±0.422</td>
<td>0.052</td>
</tr>
<tr>
<td>EKF</td>
<td>1.419±0.104</td>
<td>0.042</td>
<td>3.147±0.422</td>
<td>0.168</td>
</tr>
</tbody>
</table>

Both criteria show that there is a stabilization in the estimation performance for $P > 100$ particles. In accordance with the results presented by O’Loghlen and Wright (1982) and Souza (1992), the performance provided by the nonlinear filter and EKF are similar, except by the fact that here MCF confidence interval at $t = 1.0s$ is significantly better for $P \geq 100$, once the signal process is located around the saturation region of the observation process.

Figure 4.3 shows a typical realization of the process (1.72) embedded in the quartiles of the distribution of 100 particles obtained by the first moment of the conditional expectation. The curves in the chart suggest the symmetry of the distribution, what explains
the good results obtained by EKF.

\[
\text{FIGURE 4.4 – A realization of (4.72) and the quartiles obtained by MCF.}
\]

A comparison between MCF and EKF estimates for a typical realization of the process (4.72) is presented in Figure 4.5.

\[
\text{FIGURE 4.5 – A realization of (4.72) and its respective MCF and EKF estimates.}
\]

4.5.3.2 Unstable Case

Now the one-dimensional Ornstein-Uhlenbeck process is defined as

\[
dX_t = 0.5 X_t \, dt + 0.25 \, dW_t,
\]

where \(W_t\) is a one-dimensional standard Brownian motion and \(X_0 \sim N(0.0; 0.25)\). The corresponding observation process is as presented by equations (4.73) and (4.74).
Analogously to the last example, the performance evaluation was made based on the robustness of the pathwise MCF for \( P = 10, 100, 1000 \) and EKF estimates. This case, however, differently from that in the stable case, shows that EKF fails to provide an acceptable bounded error estimate when compared to MCF, as presented by the absolute errors \( \hat{\varepsilon} \) in Table 4.7.

**TABLE 4.7 – Estimation of \( \hat{\varepsilon}^a \) obtained by MCF and EKF at \( t = 1.0s \) and \( t = 3.0s \) for the unstable system.**

<table>
<thead>
<tr>
<th>P</th>
<th>( t = 1.0s )</th>
<th>( t = 3.0s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.703±0.081</td>
<td>1.605±0.239</td>
</tr>
<tr>
<td>100</td>
<td>0.660±0.056</td>
<td>1.436±0.115</td>
</tr>
<tr>
<td>1000</td>
<td>0.653±0.054</td>
<td>1.414±0.104</td>
</tr>
<tr>
<td>EKF</td>
<td>1.090±0.073</td>
<td>2.807±0.206</td>
</tr>
</tbody>
</table>

**TABLE 4.8 – Estimation of \( \hat{\varepsilon}^s \) obtained by MCF and EKF at \( t = 1.0s \) and \( t = 3.0s \) for the unstable system.**

<table>
<thead>
<tr>
<th>P</th>
<th>( t = 1.0s )</th>
<th>( t = 3.0s )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.801±0.173</td>
<td>4.158±1.114</td>
</tr>
<tr>
<td>100</td>
<td>0.704±0.119</td>
<td>3.357±0.575</td>
</tr>
<tr>
<td>EKF</td>
<td>1.766±0.242</td>
<td>12.328±1.718</td>
</tr>
</tbody>
</table>

Under favorable conditions the performance provided by the modified nonlinear filter and EKF were similar, in accordance with O’Loghlen and Wright (1982) and Souza (1992), but here the former obtained better results on regions where the linear approximation of the observation process was not effective. For the cases here presented, the absolute and squared error criteria appeared to be equivalent in the performance analysis, once the estimated distributions have some symmetry. Additionally, based on the example of an unstable system, it was shown that the nonlinear filter, for an adequate number of particles, can perform better than EKF, i.e. Monte Carlo-based filters may have improved performance by increasing the number of independent realizations depending on the model set point.
4.5.4 Divergence in Estimates of the Pathwise Filter

Consider the system suggested by [Davis (1981a)] and let the filtering problem be the following one-dimensional Ornstein-Uhlenbeck process

\[ dX_t = -1.0 X_t \, dt + \alpha \, dW_t \]  \hspace{1cm} (4.76)

whose one-dimensional observation process is

\[ dY_t = h(X_t) \, dt + dV_t \]  \hspace{1cm} (4.77)

with

\[ h(x) = \begin{cases} 
\sin(x) & \text{if } |x| \leq \pi/2 \\
+1 & \text{if } x > \pi/2 \\
-1 & \text{if } x < -\pi/2 
\end{cases} \]  \hspace{1cm} (4.78)

where \( W_t \) and \( V_t \) are independent one-dimensional standard Brownian motions, and \( \alpha > 0 \) is the time-independent amplitude of the state noise. The filtering was carried out for \( t \in [0, 5] \) by using a time step of \( \Delta = 2^{-8} \, s \). Given the constant \( \alpha \), the signal process (4.76) was simulated according to its analytical solution whereas the observation process (4.77) used the Euler-Maruyama scheme. The signal initialization was set to \( X_0 \sim \mathcal{N}(0.0, 0.25) \) to demonstrate the performance of the nonlinear filter over different signal to noise ratio (SNR).

Preliminary experiments performed by [Jacob and Yoneyama (2005b)] showed that the pathwise MCF started diverging when SNR in the state process is not favorable, i.e., low. The time step \( \Delta \) was decreased aiming at verifying the influence of discretization errors over the estimates. However, the error magnitudes did not change. In this way, to investigate empirically what happens to the filter, important information can be obtained by checking the average and the variance of the following coefficients: \( \epsilon \) - the estimate...
of the square error - and $\mu^\Delta$ - the normalized weight. To provide robustness in the error estimates, the analysis was made using 50 different filter realizations in 50 different initializations.

Figure 4.6 presents the results of the coefficients for the pathwise MCF given 3 different values of the noise magnitude $\alpha$: 0.10, 0.30 and 0.50. For all the values, it can be seen that the variance of the weights for $\alpha = 0.50$ is the biggest one, what suggests the filter has a threshold for stable operation with respect to the amplitude of the noise presented at the signal process. Based on these facts, the divergence effect seems to occur when the variance of $\mu^\Delta$ starts increasing. This effect suggests that a resampling scheme should be used to decrease the referred variance, making the estimates more stable.

![Graphs showing averages and variances of square error and weight](image)

FIGURE 4.6 – Averages and variances of the square error $\epsilon$ and the weight $\mu^\Delta$ for the pathwise MCF with noise amplitude given by $\alpha = 0.10, 0.30, 0.50$.

An important point to be analyzed in the robust representation is that the measure $\mathbb{F}^\Delta_P$ presented in equation (4.44) is obtained via the numerical simulation of an alternative signal process $Z^\Delta$ whose continuous-time SDE is given by

$$dZ_t = f^Y(t, Z_t)dt + \sigma(t, Z_t)dB_t,$$

where $B_t$ is a standard Brownian motion process and

$$f^Y_i(s, x) = f_i(s, x) - y(s) \sum_j g_{ij}(s, x) \frac{\partial h}{\partial x_j},$$

(4.79)
This process has a drift coefficient which depends on the observation $Y$. Assuming that the original signal process $X$ is stable, there must be specific cases where the adjusted drift $f^Y$ in the process $Z$ becomes positive, what makes the system unstable. Figure 4.7 presents the values of the average and the variance of the sampled $\bar{f}^Y_{P}$ in the conditions of the given example.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.7}
\caption{Average and variance of the adjusted drift $\bar{f}^Y_{P}$ for the pathwise MCF with noise state given by $\alpha = 0.10, 0.30, 0.50$.}
\end{figure}

Finally, the problem of the divergence in the numerical estimates of the pathwise filter occurs because the adjusted drift $\bar{f}^Y_{P}$ becomes positive and makes the discretization scheme to accumulate errors. These errors may be really significative because the estimate of $\tau_{\Delta}$ depends on the approximation of the modulation coefficient $K_n(i)$ given by (4.55), an exponential term difficult to be numerically controlled.

### 4.5.5 Generalized Pathwise Filter

Based on ideas of Del Mora (2004), a resampling scheme of the particles must be implemented in order to kill trajectories exploring unfruitful directions of the process. This can be interesting because, according to Le Gland (1984), $K_n(i)$ presented in (4.55) has a sharp maximum given a certain realization. Thus, just simulations next to the referred realization contribute effectively for the estimate.

A good criterion used to apply the resampling in a given time instant $\tau_R$ is the effective
sample size (ESS) \cite{Liu1998, Doucet1998}. This method, to be presented with more details at Chapter 6, measures the degeneracy of the particles and it can be approximated by

\[
ESS_R \approx \frac{1}{\sum_{i=1}^P (\mu^{(i)}_R)^2},
\]  

(4.80)

with the integration interval being redefined as

\[
\mu^{(i)}_R = \frac{K^{(i)}_R}{\sum_{j=1}^P \phi_1(z^{(j)}_R) K^{(j)}_R},
\]  

(4.81)

where

\[
K^{(i)}_R = \exp \left( \int_{\tau_B}^{\tau_R} c(Y, z^{(i)}_r) dr \right)
\]

(4.82)

for \(\tau_B\) being the time instant where the last resampling occurred. This scheme is just applied when \(ESS_R < ESS^{thres}\), for a given \(ESS^{thres}\) defined as a parameter of the filter.

Given an initial distribution \(\pi_0\), the step-by-step algorithm of the so-called Generalized Monte Carlo Filter (GMCF) is almost the same as the one presented by MCF, except that now the new step Replacement of particles at time \(\tau_k\) must be inserted between the steps “Importance weights evaluation” and “Conditional law computation”.

Replacement of particles

- Compute \(ESS_k\) according to equation (4.80);
- If \(ESS_k < ESS^{thres}\)
- set \(\tau_B = \tau_k\);
- For \(k = 1, ..., P\)
- replace the particles in \(z^{\Delta, (i)}_k\) proportionally to \(\mu^{(i)}_k\).
Remark 4.5.1. When the parameter $\text{ESS}^{\text{thres}}$ is chosen in a form that the resampling does not occur, the nonlinear filter becomes the traditional MCF.

Assuming the same numerical example given in the beginning of the section, a resampling scheme for $\text{ESS}^{\text{thres}} = 0.98, 0.96$ was implemented for the amplitude state noise $\alpha = 0.50$ and the results are presented in Figure 4.8 for a longer time range of simulation. The action of the resampling really makes the estimate of the filter more stable and accurate due to the control of the variance amplitude of the weights. Specifically, though a great number of resampling controls the variance amplitude given by $\text{ESS}^{\text{thres}} = 0.98$, the smoother mode of operation presented by $\text{ESS}^{\text{thres}} = 0.96$ also had good results. That is, apparently it is not necessary a great number of resamplings to control the divergence. Thus the choice of the effective number of particles is robust and its value is not the only variable to be checked in order to control effectively the stabilization of the estimates (JACOB; YONEYAMA 2005b).

![Comparison of the averages and variances](image)

FIGURE 4.8 – Comparison of the averages and variances of the square error $\epsilon$ and weights $\overline{\mu}^\Delta$ for MCF and GMCF with $\text{ESS}^{\text{thres}} = 0.98, 0.94$ and $\alpha = 0.50$.

Just the analysis of the adjusted drift $\overline{f}^{Y,P}$ can elucidate what really happens to the filter. Figure 4.9 presents the average and the variance of the given drift. The pictures show that the resampling is not decreasing effectively the variance of the drift, but inserting some bias to it. The divergence phenomenon was damped in some sense, but it is not possible to conclude at what level the numerical estimates represent the solution of the filtering problem.
FIGURE 4.9 – Average and variance of the adjusted drift $\tilde{f}_{Y,P}$ for the pathwise GMCF with noise state given by $\alpha = 0.50$ and with $ESS^{thres} = 0.98, 0.94$. 

The divergence of the pathwise MCF is a limitation of the filter, though the conclusions here presented are based on empirical results. It is well-known from the specialized literature that the approximation of unstable SDEs by numerical schemes can be rather difficult (KLOEDEN; PLATEN, 1999). For future works, a deeper study about the effects of the inclusion of this new constraint to the computation of the solution of the nonlinear filtering problem should be made from the analytical viewpoint.
5 Sequential Monte Carlo Methods

5.1 Introduction

The filtering theory was also developed assuming that the set of mathematical equations modelling a given physical system are discrete-time implemented. This approach has been extensively studied using the discrete-based Bayesian approach, a well-developed field of statistics.

In this line of research, as computing power has increased, more computationally expensive filters have been developed. These filters can be split into four basic categories. First, there are variants of EKF which obtain point estimates of the state and estimates of the mean square error of the estimators. Secondly, there are filters which approximate the posterior density of the state by mixture distributions \(\text{\textsc{sorenson; alspach, 1971}}\). Thirdly, there are grid-based filters, which the density at a set of nodes are chosen to cover the state space. This set of weighted nodes is either used as a discrete approximation \(\text{\textsc{sorenson, 1988}}\) to the posterior density, or as the basis of some continuous approximation to it, for example by splines \(\text{\textsc{kitagawa, 1987}}\). The last class of filters are those which use MC methods, and can be traced back to \(\text{\textsc{handschin and mayne, 1969}}\) and \(\text{\textsc{handschin, 1970}}\). The filters proposed there use MC methods based on the classical Bayesian theory to produce estimates for the posterior mean and covariance. More recently, the so-called sequential Monte Carlo (SMC) methods have been extensively used to produce swarm of points which approximate the whole posterior density \(\text{\textsc{gordon; salmond; smith, 1993}}\)–\(\text{\textsc{doucet; freitas; gordon, 2001b}}\).
Next sections present how MC filters have been developed using jointly the classical Bayesian theory and discretized models of physical systems. The idea is to show that this approach has a direct connection with Feynman-Kac notation [DEL MORAL 2004] used by the continuous-time filtering problem already described in Chapters 3 and 4.

5.2 Problem Statement

Given a time discretization \((\tau)_\Delta\), the continuous-time SDEs representing the mathematical modelling of physical systems can be discretized according to the traditional Euler-Maruyama or Milstein schemes [KLOEDEN; PLATEN 1999], as presented in Chapter 2. In this way, all assumptions with respect to existence and uniqueness of SDEs are the same for the discrete-time systems discussed from now on. It is assumed in this and the next chapter that discretized r.v.s are written using the same mathematical notation presented in the former chapters, i.e. \(X^\Delta = X\) hereafter.

An alternative way to define the discretized signal process \(X\) consists in embedding the random dynamical system through its transition probabilities. This approach gives some insights into methods of thinking about the evolution of the marginal laws of \(X\) and it also allows to consider signals taking values on infinite dimensional spaces. For these reasons, the sequence \(X = \{X_n\}_{n \geq 0}\) is a hidden Markov process taking values in \(\mathbb{R}^D\) with initial distribution \(p(x_0) \sim \pi_0\) and transition equation \(p(x_n|x_{n-1})\), where \(p(\cdot)\) means a pdf.

**Remark 5.2.1.** In terms of abstract representation, the sequence \(X = \{X_n\}_{n \geq 0}\) is a Markov process taking values in \(\mathbb{R}^D\) with transition probabilities \(\{K_n\}_{n \geq 1}\) defined by

\[
K_n(x_{n-1}, dx_n) = \mathbb{P}(X_n \in dx_n|X_{n-1} = x_{n-1})
\]

with initial distribution \(X_0 \sim \pi_0\).

Particularly, in accordance with the parameterization assumed in Chapters 3 and 4,
the general equation of the discrete-time signal process $X_n$ can be written as

$$X_{n+1} = \mathcal{F}_n(X_n) + \sigma_n(X_n) \Delta W_n,$$  \quad (5.1)

where: 1-) the drift function $f_n(x) = f(n, x)$ contains the adjusted drift coefficient from the original SDE in Ito or FS senses, 2-) the diffusion function $\sigma_n(x) = \sigma(n, x)$ represents the adjusted diffusion coefficient from the original SDE, and 3-) the Gaussian white noise $\Delta W_n$ means the integral of the standard Brownian motion, as described in Chapter 2. This model has transition probability given by $p(x_n|x_{n-1}) \sim \mathcal{N}(x_{n+1} - \mathcal{F}_n(x_n), \sigma^2_n(x_n))$.

The signal $X$ is not known but partially and punctually observed at the time instants established by the discretization $(\tau)\Delta$. The observation process \{Y_n\}_{n \geq 1} now refers to a sequence of $\mathbb{R}^M$-valued random variables obeying the complete model

$$Y_{n+1} - Y_n = H_n(X_n) + \Delta V_n,$$  \quad (5.2)

with $H : \mathbb{R}_+ \times \mathbb{R}^D \to \mathbb{R}^M$ being the mapping defined by

$$\forall x \in \mathbb{R}^D, \quad H_s(x) = \int_0^t h_s(x_s) ds,$$

where: 1-) the observation function $h_n(x) = h(n, x)$ contains the adjusted coefficient from the original SDE, and 2-) the Gaussian white noise $\Delta V_n$ means the integral of the non-standard Brownian motion with covariance matrix $R(n)$. This observation model has transition probability approximated by $p(y_n|x_{n-1}) \sim \mathcal{N}(y_{n+1} - y_n - H_n(x_n), R(n))$.

For an adequate transformation of variable, the observation process presented in equation (5.2) can be rewritten in a form where it becomes conditionally independent of the signal process of marginal distribution $p(y_k|x_k)$, i.e.

$$Y_{n+1} = h_n(X_n) + \Delta V_n.$$
Then \( p(y_n|x_n) \sim \mathcal{N}(y_n - h_n(x_n), R(n)) \).

**Definition 5.2.2.** The discretized model of a given physical system is a hidden Markov model (HMM) described by

\[
\begin{cases}
  p(x_0) \text{ and } p(x_n|x_{n-1}), \ n \geq 1, \\
  p(y_n|x_n) \text{ or } p(y_n|x_n,y_{n-1}), \ n \geq 0.
\end{cases}
\]

Let \( x_{0:n} \triangleq \{x_0, \ldots, x_n\} \) and \( y_{0:n} \triangleq \{y_0, \ldots, y_n\} \) be the signal and the observations up to time \( n \), respectively. The aim of the filtering is to estimate recursively in time the posterior distribution \( p(x_{0:n}|y_{0:n}) \) and its associated features including the marginal distribution \( p(x_n|y_{0:n}) \), once the solution of the problem is given by

\[
\pi_{0:n}(\varphi) = \mathbb{E}[\varphi(x_{0:n})|Y_0 = y_0, \ldots, Y_n = y_n] = \frac{\int \varphi(x_{0:n})p(x_{0:n}|y_{0:n})dx_{0:n}}{\int p(x_{0:n}|y_{0:n})dx_{0:n}}.
\]

(5.3)

The aspects of the Bayesian theory involved in the obtention of the filtering formula are discussed in the next section.

### 5.3 The Bayesian Filtering Problem

Given HMM described in Definition 5.2.2, the posterior distribution \( p(x_{0:n}|y_{0:n}) \) and its marginal distribution \( p(x_n|y_{0:n}) \) will be obtained via the traditional Bayesian theory using the assumptions of the continuous-time SDEs.

**Theorem 5.3.1 (Bayes’ Theorem).** The conditional density can be written as

\[
p(x|y) = \frac{p(y|x)p(x)}{p(y)},
\]

where the normalizing constant in the denominator can be expressed using the law of total
The function \( p(y|x) \) is labelled the likelihood while the density function of the parameters \( p(x) \) is the so-called prior. Using the total probability theorem it is worth noting that the Bayes’ law can be expressed solely using the joint density of \( x \) and \( y \),

\[
p(x|y) = \frac{p(y, x)}{\int_{\mathbb{R}^n} p(y|x)p(x)dx}.
\]

Since the conditional density is the fundamental solution to the inference problem, it follows that the joint density is the only description needed for doing statistical inference.

At time \( n \), the posterior distribution is defined as

\[
p(x_{0:n}|y_{1:n}) \xrightleftharpoons[\text{Bayes’ Theorem}]{\text{Bayes’ Theorem}} \frac{p(y_{1:n}|x_{0:n})p(x_{0:n})}{p(y_{1:n})} = \frac{p(y_{1:n}|x_{0:n})p(x_{0:n})}{\int p(y_{1:n}|x_{0:n})p(x_{0:n})dx_{0:n}},
\]

where from the definition of the Bayes’ ratio

\[
p(y_{1:n}|x_{0:n})p(x_{0:n}) = p(x_{0:n}, y_{1:n}).
\]

At time \( n + 1 \), the joint distribution is given by

\[
p(x_{0:n+1}, y_{1:n+1}) = p(y_{n+1}|x_{0:n+1}, y_{1:n})p(x_{0:n+1}, y_{1:n}),
\]

but

\[
p(x_{0:n+1}, y_{1:n}) = p(x_{n+1}|x_{0:n}, y_{1:n})p(x_{0:n}, y_{1:n})
\]
and

\[ p(x_{0:n}, y_{1:n}) = p(x_{0:n} | y_{1:n}) p(y_{1:n}). \]

Finally, the joint distribution can be rewritten as

\[ p(x_{0:n+1}, y_{1:n+1}) = p(y_{n+1} | x_{0:n+1}, y_{1:n}) p(x_{n+1} | x_{0:n}, y_{1:n}) p(x_{0:n} | y_{1:n}) p(y_{1:n}) \]

and, consequently, the conditional distribution is given by

\[
p(x_{0:n+1} | y_{1:n+1}) = \frac{p(y_{n+1} | x_{0:n+1}, y_{1:n}) p(x_{n+1} | x_{0:n}, y_{1:n}) p(x_{0:n} | y_{1:n}) p(y_{1:n})}{p(y_{1:n+1})} = \frac{p(y_{n+1} | x_{0:n+1}, y_{1:n}) p(x_{n+1} | x_{0:n}, y_{1:n}) p(x_{0:n} | y_{1:n})}{p(y_{n+1} | y_{1:n})} p(x_{0:n} | y_{1:n}) \tag{5.4}
\]

because

\[ p(y_{1:n+1}) = p(y_{n+1} | y_{1:n}) p(y_{1:n}). \]

The signal process \( X_n \) is a Markov process with transition equation \( p(x_n | x_{n-1}) \), then

\[ p(x_{n+1} | x_{0:n}, y_{1:n}) = p(x_{n+1} | x_n). \tag{5.5} \]

Applying the simplification presented in equation \( (5.3) \) into the conditional distribution \( (5.4) \), the new conditional distribution is given by

\[
p(x_{0:n+1} | y_{1:n+1}) = \frac{p(y_{n+1} | x_{0:n+1}, y_{1:n}) p(x_{n+1} | x_n) p(x_{0:n} | y_{1:n}) p(y_{1:n})}{p(y_{1:n+1})} = \frac{p(y_{n+1} | x_{0:n+1}, y_{1:n}) p(x_{n+1} | x_n) p(x_{0:n} | y_{1:n})}{p(y_{n+1} | y_{1:n})} p(x_{0:n} | y_{1:n}). \tag{5.6}
\]

To finish the filtering problem, the conditional distribution \( (5.6) \) must be adjusted the observation models \( p(y_n | x_n) \) and \( p(y_n | x_n, y_{n-1}) \) presented in Definition \( (5.2.2) \).
For the first observation case \( y_i \) is conditionally independent given \( x_i \), and has marginal distribution \( p(y_i|x_i) \), what implies in

\[
p(y_{n+1}|x_{0:n+1}, y_{1:n}) = p(y_{n+1}|x_{n+1})
\]

and then to the final format of the conditional distribution

\[
p(x_{0:n+1}|y_{1:n+1}) = \frac{p(y_{n+1}|x_{n+1}) p(x_{n}|x_{n+1})}{p(y_{n+1}|y_{1:n})} p(x_{0:n}|y_{1:n}).
\]

Analogously, for the second observation case,

\[
p(y_{n+1}|x_{0:n+1}, y_{1:n}) = p(y_{n+1}|x_n, y_n)
\]

and then

\[
p(x_{0:n+1}|y_{1:n+1}) = \frac{p(y_{n+1}|x_n, y_n) p(x_{n+1}|x_n)}{p(y_{n+1}|y_{1:n})} p(x_{0:n}|y_{1:n}).
\]

This recursion is only academic in the sense that one cannot typically compute the normalizing constant \( p(y_{0:n+1}) \), the marginals of \( p(x_{0:n+1}, y_{0:n+1}) \) and in particular \( p(x_{n+1}|y_{n+1}) \). The idea is to assume that it is possible to sample according to \( p(x_n|x_{n-1}) \) and that \( p(x_n|x_{n-1}) \) and \( p(y_n|x_n) \) can be evaluated pointwise.

### 5.4 Sequential Monte Carlo Filters

Particle filters are SMC methods which approximate the posterior densities by swarms of points, the so-called \textit{particles}, in the sample space. The particles each have an assigned weight and the posterior distribution can then be approximated by a discrete distribution which has support on each of the particles. The probability assigned to each particle is proportional to the weight.
The algorithms differ in the way that the swarm of particles evolves and adapts to incoming data. Some similarities can be seen between particle filters and grid-based methods (Bergman, 1999), once they both produce a set of weighted points and use these as a basis for an approximation of the posterior density. However, while the points of the grid based filters are chosen arbitrarily by the user, and a new choice may have to be made each time step so that the grid follows the moving state, particles are generated randomly from the system equation, and naturally follow the movement of the state. Further, the computation involved in calculating the weights of the particles is considerably smaller than the computation required to calculate the weights assigned to each grid point for the grid based filters. An algorithm of this type, a SMC method called Bayesian bootstrap filter, was first suggested by Gordon, Salmond and Smith (1993) and after by Lin and Chen (1995) and Kitagawa (1996). Since then a number of similar algorithms have been proposed in such fields as computer science, engineering and statistics. Recently, alternative algorithms have been proposed with different rules for propagating the swarm of points, as presented by Doucet, Freitas and Gordon (2001b) and Ristic, Arulampalam and Gordon (2004).

5.4.1 Perfect Monte Carlo sampling

The solution of the filtering problem presented in equation (5.3) can be rewritten using the measure theory as

\[ \pi_{0:n}(\varphi) = \frac{\int \varphi(x_{0:n}) \mathbb{P}^\Delta(dx_{0:n}|y_{0:n})}{\int \mathbb{P}^\Delta(dx_{0:n}|y_{0:n})}, \]  

(5.7)

a clearly representation of Feynman-Kac formula (Del Moral, 2004).

Assuming the possibility to simulate \( P \) i.i.d. random samples \( \{x_{0:n}^{(i)}; i = 1, \ldots, P\} \) according to \( p(x_{0:n}|y_{0:n}) \), an empirical estimate of the measure \( \mathbb{P}^\Delta, P \) can be given by:

\[ \mathbb{P}^\Delta, P(dx_{0:n}|y_{0:n}) = \frac{1}{P} \sum_{i=1}^{P} u_n^{(i)} \delta_{x_{0:n}^{(i)}}(dx_{0:n}), \]  

(5.8)
where $\delta_{i}^{(i)} (dx_{0:n})$ denotes the delta-Dirac mass located in $x_{0:n}^{(i)}$ and $w_{n}^{(i)}$ corresponds to all information with respect to the particle $i$. One obtains straightforwardly the following estimate of $\pi_{0:n}(\varphi)$:

$$I_{P}(\varphi) = \int \varphi(x_{0:n}) \mathbb{P}_{X_{0:n}|y_{0:n}}(dx_{0:n}|y_{0:n}) = \frac{1}{P} \sum_{i=1}^{P} \varphi(x_{0:n}^{(i)}) w_{n}^{(i)}. \quad (5.9)$$

From the strong law of large numbers

$$I_{P}(\varphi) \xrightarrow{a.s.} \pi_{0:n}(\varphi).$$

If the posterior variance of $\varphi(x_{0:n})$ satisfies

$$\sigma_{\varphi}^{2} \triangleq \text{var}_{p(\cdot|y_{0:n})} [\varphi(x_{0:n})] = \mathbb{E}_{p(\cdot|y_{0:n})} [\varphi^{2}(x_{0:n})] - \mathbb{E}_{p(\cdot|y_{0:n})} [\varphi(x_{0:n})]^{2} < \infty, \quad (5.10)$$

then a central limit theorem holds for convergence in distribution

$$\sqrt{P} [I_{P}(\varphi) - \pi_{0:n}(\varphi)] \xrightarrow{P \rightarrow \infty} \mathcal{N}(0, \sigma_{\varphi}).$$

The advantage of this perfect MC method is clear. From the given set of r.v.s $\{x_{0:n}^{(i)}\}_{i=1}^{P}$, one can easily estimate any quantity $\mathbb{E}_{p(\cdot|y_{0:n})} [\varphi(x_{0:n})]$ and the convergence speed of this estimate neither depends on $Dx n$ nor on $\varphi(x_{0:n})$, i.e. independent of the dimension of the integrand, but only on $P$. Any deterministic numerical integration method has a convergence rate that decreases as the dimension of the integrand increases. Unfortunately, it is usually impossible to sample efficiently from the posterior distribution $p(x_{0:n}|y_{0:n})$ at any time $n$, being $p(x_{0:n}|y_{0:n})$ multivariate, non standard and only known up to a proportionality constant (DOUCET, 1998).
5.4.2 Bayesian Importance Sampling

MC framework for numerical integration and optimization rests on the assumption that there exists $P \gg 1$ samples easily obtained in practice from a generic density $\pi(x)$. For standard distributions such as uniform, Gaussian, Gamma, Student, etc., several perfect random sampling algorithms exist. Uniform i.i.d. r.v.s can be generated by some pseudo random sequence with very long repetition time. Other standard distributions are generally obtained by feeding a possibly approximative, but often exact, inverse of the cumulative distribution function with a pseudo random sequence. Higher dimensional r.v.s and more general distributions can be generated by combinations and mixtures of basic distributions (RIPLEY, 1988) for a thorough treatment of random number generation.

Usually, the density under consideration is not a familiar combination or mixture of the basic distributions, and it is not possible to directly generate samples from $\pi(x)$ presented by $\pi_{0:n}(\varphi)$ in equation (5.7). When there is a known upper bound on the density function values, and it is possible to evaluate the density pointwise, it is still possible to generate samples from $\pi(x)$. The rejection sampling procedure, to be presented in Chapter 6, attains this although rather inefficiently. When it is possible to generate samples from a density similar to the desired one, a correct weighting of the sample set makes MC estimation possible. The importance sampling methods assume that it is possible to evaluate $\pi(x)$ up to a normalization constant, and that the proposal distribution covers the support of $\pi(x)$.

Importance sampling also deals with a proposal distribution $q(x)$ which is easy to generate samples from. However, the only general assumption on the importance function $q(x)$ is that its support set covers the support of $\pi(x)$, i.e.

$$\pi(x) > 0 \Rightarrow q(x) > 0 \text{ for all } x \in \mathbb{R}^D.$$
Under this assumption,

$$I = \int_{\mathbb{R}^D} \varphi(x) \pi(x) \, dx = \int_{\mathbb{R}^D} \frac{\varphi(x)}{q(x)} q(x) \, dx. \tag{5.11}$$

A MC estimate is computed by generating $P \gg 1$ independent samples from $q(x)$, and forming the weighted sum

$$\varphi_P = \frac{1}{P} \sum_{i=1}^{P} \varphi(x^{(i)}) w(x^{(i)}), \text{ with } w(x^{(i)}) = \frac{\pi(x^{(i)})}{q(x^{(i)})}, \tag{5.12}$$

where $w(x^{(i)})$ is the so-called importance weight.

If the normalizing factor of the target density $\pi(x)$ is unknown, the importance weights in (5.12) can only be evaluated up to a normalizing factor. Then, the weights can be formed using a function proportional to the target density and then normalized afterwards, forming the estimate

$$\varphi_P = \frac{\sum_{i=1}^{P} \varphi(x^{(i)}) w^{(i)}}{\sum_{i=1}^{P} w(x^{(i)})}, \text{ where } w^{(i)} \propto \frac{\pi^{(i)}}{q^{(i)}}. \tag{5.13}$$

This technique is practically applied in the Bayesian framework and therefore often referred to as Bayesian importance sampling. The estimate (5.13) is biased for finite $P$, but asymptotically both a law of large numbers and a central limit theorem hold.

**Theorem 5.4.1 (Geweke (1989)).** When equation (5.11) exists and is finite, the estimate (5.13) converges almost everywhere

$$\mathbb{P}(\lim_{P \to \infty} \varphi_P = I) = 1.$$

Additionally, if $\mathbb{E}[w(x)] < \infty$ and $\mathbb{E}[\varphi^2(x) w(x)] < \infty$

$$\lim_{P \to \infty} \sqrt{P} (\varphi_P - I) \sim \mathcal{N}(0, \sigma^2),$$

where $\sigma^2 = \mathbb{E}[(\varphi(x) - I)^2 w(x)]$. All expectations are performed w.r.t. the density $\pi(x)$. 
The importance sampling procedure yields a MC estimate of integral (5.11). The Sampling Importance Resampling (SIR) algorithm of [Rubin] (1987) is a procedure for generating an approximately independent draw from $\pi(x)$ using its weighted approximation. The independent draw from $\pi(x)$ is obtained by inserting a resampling step after the weight calculations.

### 5.4.2.1 The SIR Algorithm

The basic idea is to simulate the samples $\{x_{0:n}^{(i)}\}$ not directly from $p(x_{0:n}|y_{0:n})$, but rather from an alternative proposal density, or importance sampling distribution, $q(x_{0:n}|y_{0:n})$. Assuming that the integral $I(\varphi)$ must be evaluated, and provided that the support of $q(x_{0:n}|y_{0:n})$ includes the support of $p(x_{0:n}|y_{0:n})$, the identity given in equation (5.7) is obtained, but typically the denominator cannot be expressed in closed form, then

$$I(\varphi) = \frac{\int \varphi(x_{0:n}) \omega(x_{0:n}) q(x_{0:n}|y_{0:n}) dx_{0:n}}{\int \omega(x_{0:n}) q(x_{0:n}|y_{0:n}) dx_{0:n}}, \quad (5.14)$$

where $\omega(x_{0:n})$ is the unnormalized importance weight

$$\omega(x_{0:n}) = \frac{p(x_{0:n}|y_{0:n})}{q(x_{0:n}|y_{0:n})} \propto \omega_n^{(i)}.$$

Consequently, if one can simulate $P$ i.i.d. particles according to $q(x)$, a possible MC estimate of $I(\varphi)$, as an intuitive approximation, is given by the ratio of the estimates of the numerator and denominator obtained, i.e.

$$I(\varphi) = \hat{I}_P(\varphi) = \frac{P^{-1} \sum_{i=1}^{P} \varphi(x_{0:n}^{(i)}) \omega(x_{0:n}^{(i)})}{P^{-1} \sum_{i=1}^{P} \omega(x_{0:n}^{(i)})} = P^{-1} \sum_{i=1}^{P} \varphi(x_{0:n}^{(i)}) \tilde{\omega}(x_{0:n}^{(i)}),$$

where the normalized importance weights are given by

$$\tilde{\omega}_n^{(i)} = \frac{\omega(x_{0:n}^{(i)})}{\sum_{i=1}^{P} \omega(x_{0:n}^{(i)})}.$$
The true importance weights $\omega_n^{s(i)}$ have been replaced by the estimate $\tilde{\omega}_n^{s(i)} = P^{\hat{\omega}_n^{s(i)}}$.

Importance sampling is a general MC integration method. In its simplest form, it is not adequate for recursive estimation. The computational complexity of this operation increases with time because, at each new data $y_{n+1}$ one needs to recompute the importance weights over the entire state sequence.

The importance sampling method can be modified so that it becomes possible to compute an estimate $P^\Delta P(dx_{0:n} | y_{1:n})$ of $p(x_{0:n} | y_{0:n})$ without modifying the past simulated trajectories $\{x_0^{(i)} \}_{i=1}^P$. This means that $q(x_{0:n} | y_{0:n})$ at time $\tau_n$ admits as marginal distribution at time $\tau_{n-1}$ the importance function $q(x_{0:n-1} | y_{0:n-1})$, i.e.

$$q(x_{0:n} | y_{0:n}) = q(x_{0:n-1} | y_{0:n-1})q(x_n | x_{0:n-1}, y_{0:n})$$

and, finally,

$$q(x_{0:n} | y_{0:n}) = q(x_0 | y_{0:n}) \prod_{k=1}^n q(x_k | x_{0:k-1}, y_{0:n}).$$

It is easy to see that this importance function allows one to evaluate recursively in time the normalized and unnormalized importance weights.

It follows then that

$$\hat{\omega}(x_{0:k}^{(i)}) = \frac{p(x_{0:n}^{(i)} | y_{0:n})}{q(x_{0:n}^{(i)} | y_{0:n})} = \frac{1}{q(x_{0:n}^{(i)} | y_{0:n})} \frac{p(y_{n} | x_{0:n}^{(i)}, y_{1:n-1})p(x_{0:n}^{(i)} | x_{0:n-1}^{(i)}, y_{1:n-1})p(x_{0:n-1}^{(i)} | y_{1:n-1})}{p(y_{n} | y_{0:n-1})} = \frac{1}{p(y_{n} | y_{0:n-1})} \frac{p(x_{0:n-1}^{(i)} | y_{1:n-1})p(y_{n} | x_{0:n}^{(i)}, y_{1:n-1})p(x_{0:n-1}^{(i)} | x_{0:n-1}^{(i)}, y_{1:n-1})}{q(x_{0:n}^{(i)} | y_{0:n-1})} = \frac{1}{p(y_{n} | y_{0:n-1})} \hat{\omega}(x_{0:n-1}^{(i)})q(x_{0:n}^{(i)} | y_{1:n}),$$
where

\[ \hat{\omega}(x_{0:n-1}^{(i)}) = \frac{p(x_{0:n-1}^{(i)}|y_{1:n-1})}{q(x_{0:n-1}^{(i)}|y_{1:n-1})} \]

and

\[ \gamma(x_{0:n}^{(i)}, y_{1:n}) = \frac{p(y_n|x_{0:n}^{(i)}, y_{1:n-1})p(x_n^{(i)}|x_{0:n-1}^{(i)}, y_{1:n-1})}{q(x_n^{(i)}|x_{0:n-1}^{(i)}, y_{1:n})}. \]

Normalizing the weights \( \hat{\omega}(x_{0:n-1}^{(i)}) \), one obtains

\[ \omega(x_{0:n}^{(i)}) = \frac{\hat{\omega}(x_{0:n-1}^{(i)})}{\sum_{j=1}^{P} \hat{\omega}(x_{0:n-1}^{(j)})} = \frac{\hat{\omega}(x_{0:n-1}^{(i)})\gamma(x_{0:n}^{(i)}, y_{1:n})}{\sum_{j=1}^{P} \hat{\omega}(x_{0:n-1}^{(j)})\gamma(x_{0:n}^{(j)}, y_{1:n})}. \] (5.15)

But,

\[ \hat{\omega}(x_{0:n-1}^{(i)}) = \left[ \sum_{m=1}^{P} \hat{\omega}(x_{0:n-1}^{(m)}) \right] \omega_{n-1}^{(i)}. \] (5.16)

Replacing equation (5.16) into (5.15) and following the notation developed by Pavlov (2004), one gets

\[ \omega_n^{(i)} = \frac{\hat{\omega}(x_{0:n-1}^{(i)})\gamma(x_{0:n}^{(i)}, y_{1:n})}{\sum_{j=1}^{P} \left[ \sum_{m=1}^{P} \hat{\omega}(x_{0:n-1}^{(m)}) \right] \omega_{n-1}^{(j)}\gamma(x_{0:n}^{(j)}, y_{1:n})} \]

\[ = \frac{\hat{\omega}(x_{0:n-1}^{(i)})\gamma(x_{0:n}^{(i)}, y_{1:n})}{\sum_{m=1}^{P} \hat{\omega}(x_{0:n-1}^{(m)}) \sum_{j=1}^{P} \omega_{n-1}^{(j)}\gamma(x_{0:n}^{(j)}, y_{1:n})} \]

\[ = \frac{\omega_n^{(i)}\gamma(x_{0:n}^{(i)}, y_{1:n})}{\sum_{j=1}^{P} \omega_{n-1}^{(j)}\gamma(x_{0:n}^{(j)}, y_{1:n})}. \]

Finally the weight is given by

\[ \omega_n^{(i)} = B_n w_n^{(i)}\gamma(x_{0:n}^{(i)}, y_{1:n}), \] (5.17)

where \( B_k \) is computed such that \( \sum_{j=1}^{P} \omega_n^{(j)} = 1. \)
The implementation of these filters will be described in Chapter 6, where particle selection schemes are developed attempting to improve the filtering performance.
6 Particle Selection Schemes

6.1 Introduction

Particle filters can be defined as SMC methods used to solve estimation problems where time-varying signals must be presented in real-time. Based on the parametric structure of a probabilistic dynamic system \cite{LIU; CHEN; LOGVINENKO, 2001}, these problems are described by the estimation of non-observable states of the model and/or detection of events described by the observed signals.

More specifically, if a SMC method implements a recursive Bayesian filter by using MC simulations, this new method is called sequential importance sampling (SIS) algorithm \cite{ARULAMPALAM et al, 2002}. The estimates are computed from the representation of the required posterior density function via a set of importance weighted random samples. As the number of samples becomes very large, this representation tends to an equivalent one of the usual functional description of the posterior pdf.

The problem encountered by SIS method is that, as time goes by, the distribution of the weighted random samples becomes more and more skewed, the so-called degeneracy phenomenon \cite{DOUCET; GODSILL; ANDRIEU, 2000}. In this way, a resampling step must be added to the algorithm aiming at minimizing this problem, what means that it crucially affects the overall particle filter performance \cite{GORDON; SALMON; SMITH, 1993, LIU; CHEN, 1995}. Specifically, if the resampling step is applied at every algorithm iteration, the so-called sequential importance resampling (SIR) method, or bootstrap, is obtained \cite{GODSILL; CLAPP, 2001}.
In practical applications of particle filters, a large number of particles needs to be used for computing the estimates of desired states. However, the possibility of parallelization of the filter is affected by the resampling that has the following disadvantages from the hardware implementation viewpoint: 1-) the sampling period and memory requirements are increased, and 2-) the data exchange in implementations with multiple processing elements becomes a bottleneck of the parallel design [BOLIC 2004]. In this way, modified resampling algorithms must be proposed to have an efficient mechanism that reduces the hardware complexity and maintains the filter performance [HONG; BOLIC; DJURIĆ 2004].

The weight update and particles diffusion can straightforwardly be calculated independently. The resampling is, however, not possible to parallelize and this is also the part that is most time-consuming. Therefore it is of most importance that a good proposal is chosen to minimize the frequency of resampling. One way to achieve a parallel implementation is to run several particle filters in parallel [DE FREITAS 1999]. A very high degree of parallelism cannot be achieved since each filter must have a rather large sample size $P$, so that the particle cloud of each filter resembles the posterior and the estimates do not diverge. The state estimates are based on a MC average over the total set of all points in all filters, with the weights appropriately normalized so that all the weights sum to unity.

The numerical complexity of SIS algorithm is $O(P)$, what in practice is important as $P \gg 1$. In the general case, the memory requirements are $O((n + 1)P)$ as it is necessary to keep all the $P$ simulated trajectories from time 0 to time $n$. However, if $q(x_n|x_{0:n-1}^{(i)}, y_{0:n})$ and if one is only interested in the filtering distribution $p(x_n|y_{0:n})$, the memory requirements are $O(P)$.

This chapter proposes a method to address the particle degeneracy phenomenon by using the restoration of particles [JACOB; YONEYAMA 2005a]. The idea is to replace the resampling by an open alternative method that moves the particles in the likelihood estimate to regions where the weights can be higher. The implementation of this new method allows the possibility of parallelization of the algorithm and the particle filter performance can be similar to that obtained by SIR method.
6.2 The Generic Bootstrap Approach

At every time instant $k$, let $\{x_{1:k}^{(i)}, w_k^{(i)}\}_{i=1}^P$ denote a random measure where $x_{1:k}^{(i)}$ is the $i$th particle of the signal at $k$, $x_k^{(i)}$ is the $i$th trajectory of the signal, and $w_k^{(i)}$ is the weight of the $i$th particle. It is supposed that this random measure characterizes the posterior

$$p(x_k|y_{1:k}) \approx \sum_{i=1}^P w_k^{(i)} \delta(x_k - x_k^{(i)}), \quad (6.1)$$

where $y_{1:k}$ is the observation signal up to $k$ and $w_k^{(i)}$ is normalized. Therefore, equation (6.1) is a discrete weighted approximation to the true posterior $p(x_k|y_{1:k})$, where the weights are chosen using the principle of importance sampling (Doucet, 1998).

According to the classical SIS (Doucet; Godsill; Andrieu, 2000), the weight updating for the optimal importance density function that minimizes the variance of the weights is given by

$$w_k^{(i)} \propto w_{k-1}^{(i)} \frac{p(y_k|x_k^{(i)})p(x_k^{(i)}|x_{k-1}^{(i)})}{p(x_k^{(i)}|x_{k-1}^{(i)}, y_k)} = w_{k-1}^{(i)}p(y_k|x_{k-1}^{(i)}), \quad (6.2)$$

where $p(y_k|x_{k-1}^{(i)})$ does not have an analytical expression in the general case. To facilitate the filter implementation (Gordon; Salmond; Smith, 1993)-(Kitagawa, 1996), it is assumed that $p(x_k^{(i)}|x_{k-1}^{(i)})$ is a particular choice of importance density, what implies in a weight updating given by

$$w_k^{(i)} \propto w_{k-1}^{(i)} p(y_k|x_k^{(i)}). \quad (6.3)$$

An important point is that this chosen importance sampling density is independent of $y_k$ and, therefore, the state space is explored without any knowledge of the observations.

Due to the particle degeneracy phenomenon (Doucet; Godsill; Andrieu, 2000), a new step must be added to the algorithm in order to address this problem and also to improve the filter performance (Andrieu; Doucet; Punskaya, 2001)-(Liu; Chen; Log STL;...
as described in the following generic algorithm for the time instant $\tau_k$:

**Generic SIS Filter**

- For $i = 1, \ldots, P$
  - sample $x_k^{(i)} \sim p(x_k | x_{k-1}^{(i)})$;
  - evaluate $w_k^{(i)} = p(y_k | x_k^{(i)})$.
- Evaluate $w_k^T = \sum_{j=1}^P w_k^{(j)}$.
- For $i = 1, \ldots, P$
  - evaluate $w_k^{(i)} = (w_k^{(i)})^{-1} w_k^T$.
- Apply a method to minimize the degeneracy phenomenon if it exists
  - obtain $\{\hat{x}_k^{(i)}, \hat{w}_k^{(i)}\}_{i=1}^{\hat{P}}$, with $\hat{P} \leq P$.

Based on the heuristic that particles having higher weights must be duplicated and others must be discarded, the resampling method (Gordon; Salmond; Smith, 1993; Liu; Chen, 1995) is used to minimize the particle degeneracy problem. This method aims at obtaining an unweighted empirical distribution approximation of $p(x_k | y_{1:k})$ presented in equation (6.1) (Crisan; Doucet, 2002; Doucet; Freitas; Gordon, 2001b), in accordance with the properly weighted sample principle (Liu; Chen; Logvinenko, 2001). When the resampling procedure is applied at each time instant $k$, the bootstrap filter is obtained.

### 6.3 SIS principle

MC filters (MCF) can be loosely defined as a set of methods that use MC simulation to solve on-line estimation and prediction problems in dynamic systems, as presented by Liu, Chen and Logvinenko (2001). Compared with traditional filtering methods, simple, flexible - yet powerful - MCF techniques provide effective means to overcome computa-
tional difficulties in dealing with nonlinear dynamic models. One key element of MCF techniques is the recursive use of resampling principle, which leads to the more precise name SIS for the techniques that are the focus of this chapter.

This section introduces the basic concept of weighted samples in the context of SIS methods. Then, based on this concept, some major improvements for enhancing SIS algorithms are presented, such as the resampling [GORDON; SALMOND; SMITH 1993], the systematic resampling (SR) [FEARNHEAD 1998], the reweighting and the reallocation (PITT; SHEPHARD 1999), and the rejection control (RC) [LIU 1996]–[LIU; CHEN 1998].

### 6.3.1 Properly Weighted Sample

Suppose one wants to evaluate \( \theta = \mathbb{E}_\pi h(x) \) for some arbitrary function \( h \) under the target distribution \( \pi(x) \) given up to a normalizing constant.

**Definition 6.3.1.** A set of weighted random samples \( \{(x^{(i)}, w^{(i)})\}_{i=1}^P \) is called proper with respect to \( \pi \) if, for any square integrable function \( h(\cdot) \),

\[
\mathbb{E} [h(x^{(i)})w^{(i)}] = c \mathbb{E}_\pi h(x), \text{ for } i = 1, \ldots, P,
\]

where \( c \) is a normalizing constant common to all the \( P \) samples.

With this set of weighted samples, it is possible to estimate \( \theta \) as

\[
\hat{\theta} = \frac{1}{W} \sum_{i=1}^P w^{(i)} h(x^{(i)}),
\]

where \( W = \sum_{j=1}^P w^{(j)} \). For instance, if the \( x^{(i)} \) are drawn from \( \pi \) directly, the set of \( \{(x^{(i)}, 1)\} \) is proper. But in the context of importance sampling, one draws \( x^{(i)} \) from a trial distribution \( q(x) \) and gives it a weight

\[
w^{(i)} = \frac{\pi(x^{(i)})}{q(x^{(i)})}.
\]
Then \( \{(x^{(i)}, w^{(i)})\} \) is proper with respect to \( \pi \). One reason for using the renormalised estimates given by \( \hat{\theta} \) in an importance sampling framework is that one does not need to know the normalizing constant for \( \pi \), \( i.e. \) in Definition \( 6.3.1 \) the constant \( c_P \) can be unknown.

MC’s view of the world is that any probability distribution \( \pi \), regardless of its complexity, can always be represented by a discrete MC sample from it. It means that any computation of expectations using \( \pi \) can be replaced to an acceptable degree of accuracy by using the empirical distribution resulting from the discrete sample. When dealing with importance sampling, \( \pi \) can be represented, at least conceptually, by any set of properly weighted samples. An important issue in applications, however, is how to find a convenient and efficient discrete representation, \( i.e. \) the representation set that is easily generated whose estimation is accurate.

### 6.3.2 Operations for enhancing SIS

The degeneracy phenomenon of the weights difficulties the implementation of the filters, once it is necessary to increase the number of particles to compensate the lack of accuracy when, in some sense, the weights do not represent adequately the likelihood properly.

Basically, the concept of reorganizing particles means that all the information contained in the weights will be distributed to the new particle location. This reorganization makes the particles independent. As the time goes by, the weights absorb the information from the past again and the particles become correlated, what causes the degeneracy phenomenon from the empirical standpoint.

Next sections present four important operations to minimize the given phenomenon, what makes SIS feasible to operate in practical applications.
6.3.2.1 Resampling

Suppose at time instant $t_k$ there exists a set of random samples $\Psi_k = \{(x_k^{(i)}, w_k^{(i)})\}_{i=1}^P$ properly weighted with respect to $\pi$. By treating $\Psi_t$ as a discrete representation of $\pi$, it is possible to generate another discrete representation $\hat{\Psi}_k = \{(\tilde{x}_k^{(j)}, \tilde{w}_k^{(j)})\}_{j=1}^\hat{P}$, $\hat{P} < P$, approximately proper w.r.t. $\pi$ (RUBIN 1987) by using the following algorithm:

- For $j = 1, \ldots, P$,
  - let $\tilde{x}_k^{(j)}$ be $x_k^{(i)}$ independently with probability proportional to $\kappa^{(j)}$;
  - let the new weight of $\tilde{x}_k^{(j)}$ be $\tilde{w}_k^{(j)} = w_k^{(j)}/\kappa^{(j)}$.

- Return the new representation $\hat{\Psi}_k = \{(\tilde{x}_k^{(j)}, \tilde{w}_k^{(j)})\}_{j=1}^\hat{P}$ with $\hat{P} < P$.

It is not obvious why resampling is useful. In fact, it does not help at all in a static importance sampling scheme. A few heuristics (LIU; CHEN 1995) are as follows:

- resampling can prune away hopelessly unrepresentative samples by giving them a small $\kappa^{(j)}$;
- resampling can produce multiple copies of good samples by giving them a big $\kappa^{(j)}$ to help generate better future samples in SIS setting.

Consequently, in a probabilistic dynamic system with SIS, resampling helps on steer towards the right direction. In light of these arguments, one should choose $\kappa^{(j)}$ as a monotone function of $w_k^{(j)}$.

If $\kappa^{(j)} = w_k^{(j)}$ is taken, then the foregoing scheme is exactly the same as the one described by Gordon, Salmond and Smith (1993) and Liu and Chen (1995). This is the so-called traditional resampling.
6.3.2.2 Systematic Resampling

There are a number of useful methods for reducing variance in MC integration, as presented by Hammersley and Handscomb (1964) and Fishman (1996). For the context of particle filters, the main result comes from the survey sampling theory.

Theorem 6.3.2 (Cochran (1963)). Let \( \Theta \) be a r.v. with pdf \( p(\theta) \). Assume that this density has the mixture representation

\[
p(\theta) = \sum_{i=1}^{M} \beta_i p_i(\theta).
\]

The most efficient MC estimator of some function of \( \Theta \), \( \Gamma(\Theta) \), based on a sample of total size \( P \), with \( P_i \) samples from each \( p_i(\theta) \), is the Neymann allocation with fixed size \( P_i \propto \beta_i \sigma_i \). Here \( \sigma_i^2 \) denotes the variance of \( \Gamma(\Theta) \) under the density \( p_i(\theta) \).

The application of this theorem is given in SIS filter using SR. A further problem with applying this theorem is that in general the numbers \( P_i = P \beta_i \) will not all be integers. The best that can be achieved, without resorting to carrying forward weights, is to let \( P_i \) be a r.v. whose expected value is \( P \beta_i \), and which takes the value of either \( \lfloor P \beta_i \rfloor \) or \( \lfloor P \beta_i \rfloor + 1 \), where \( \lfloor \rfloor \) denotes the integer part of \( x \). The \( P \)'s are also constrained so that their sum is \( P \) (Fearnhead, 1998).

The algorithm for generating an array of labels \( \{l^{(m)}\}_{m=1}^{P} \) or new positions for the random samples \( \Psi_k = \{(x_k^{(i)}, w_k^{(i)})\}_{i=1}^{P} \) at time \( \tau_k \) is given as follows, in accordance with SR principle:

---

Systematic Resampling

**Initialization**

- Sample \( \lambda \sim \mathcal{U}[0, \frac{1}{P}] \);
- Set \( i = 1, j = 1 \) and \( \Upsilon = w_k^{(i)} \).
Selection

- Do While ($\lambda < 1$)
  - If ($\Upsilon > \lambda$) then
    * $\lambda \leftarrow \lambda + \frac{1}{\beta}$;
    * $l^{(i)} \leftarrow x^{(i)}_k$;
    * Set $j = j + 1$.
  - else
    * Set $i = i + 1$;
    * $\Upsilon \leftarrow \Upsilon + w^{(i)}_k$.

6.3.2.3 Reweighting and Reallocation

Having additional flexibility in choosing the sampling weights $\kappa^{(j)}$ is rather intriguing and can potentially be very useful. For example, the $\kappa^{(j)}$ can be chosen to reflect certain “future trend”, the so-called auxiliary particle filter [PITT; SHEPHARD, 1999], or be chosen to balance the need of diversity with the need of focus, the so-called Monte Carlo Markov chain (MCMC) methods [GILKS; RICHARDSON; SPIEGELHALTER, 1996]. Suggestions for $\kappa^{(j)}$ are $\sqrt{w^{(j)}_k}$ or $[w^{(j)}_k]^\alpha$, where $\alpha$ can vary according to the variation coefficient of $w^{(j)}_k$.

An important point regarding the generation of $\tilde{\Psi}_k$ is that the extra variation due to resampling is unnecessary and unwanted [LIU; CHEN, 1995]. Instead of resampling, a more efficient approach is the partially deterministic reallocation. The following scheme can be implemented for generating $\tilde{\Psi}_k$ from $\Psi_k$:

- For $i = 1, \ldots, P$
  - If $\kappa^{(i)} \geq 1$ then
    * retain $N = \lfloor \kappa^{(i)} \rfloor$ copies of the sample $x^{(i)}_k$;
    * assign weight $\tilde{w}^{(j)}_k = \frac{w^{(j)}_k}{N}$ for each copy.
- else
  * remove the sample with probability \(1 - \kappa^{(j)}\);
  * assign weight \(\tilde{w}_k^{(j)} = w_k^{(j)}/\kappa^{(j)}\) to the survived sample.
• Return \(\tilde{\Psi}_k\).

This scheme tends to slightly decrease the total sample size. Alternatively, one can choose \(N = [\kappa^{(j)}] + 1\), which will tend to slightly increase the sample size. In order to maintain a fixed sample size, a residual-resampling strategy \cite{LiuChen1993} can be applied to make up the lost samples.

### 6.3.2.4 Rejection Control and Partial Rejection Control

Another useful technique for rejuvenating the particles is RC method \cite{LiuChen1998}, which can be understood as a combination of the rejection method \cite{VonNeumann1951} and importance sampling. In RC, one monitors the coefficient of variation (CV) of the importance weights for \(x_k^{(i)}\) defined as

\[
CV_k^2 = \frac{\text{var}(w_k^{(i)})}{\text{E}^2(w_k^{(i)})}, \tag{6.4}
\]

This \(CV^2\) can be used to derive a heuristic criterion, i.e. the effective sample size:

\[
ESS_k = \frac{P}{1 + CV_k^2}, \tag{6.5}
\]

which is heuristically understood as the equivalent number of i.i.d. samples at time \(\tau_k\) \cite{Liu1996}. Once \(ESS_k\) drops below a threshold, say \(ESS_k \leq \alpha_0 P, 0 < \alpha_0 < 1\), one call this time a dynamic check-point. The check-point sequence prescribed in advance, e.g. every 10 steps, will be called static check-points.

When encountering a new check-point at time \(\tau_m\), i.e. the \(m\)-th check-point, one
computes a control threshold $\mu_m$ that may be a quantity given in advance or the median or quantile of the $w_m^{(i)}$. Then one checks every sample to decide whether to accept it according to probability $\min\{1, w_m^{(i)}/\mu_m\}$. In other words, those samples with weights greater than or equal to $\mu_m$ are automatically accepted, whereas those with weights less than $\mu_m$ are accepted with probability. All accepted samples are given a new weight $w_m^{(j)} = \max\{\mu_m, w_m^{(i)}\}$. All rejected ones are restarted from $\tau_0$ and rechecked at all previous check-points. It has been shown [LIU; CHEN [1998] that RC operation is proper in the sense of MC estimation and it always increases ESS. A problem with RC is that its computation cost increases rapidly as $k$ increases, although the threshold $\mu_k$ can be adjusted by other user to compromise between this cost and ESS.

6.4 The Restoration Method

The generic SIS filter states that the weight associated to a particle $x_k^{(i)}$ sampled from $p(x_k^{(i)}|x_{k-1}^{(i)})$ is computed by $w_k^{(i)} = p(y_k|x_k^{(i)})$. The restoration method consists in moving the particles in the obtained likelihood estimate $\{x_k^{(i)}, w_k^{(i)}\}_{i=1}$ towards regions where the importance weights are higher, in accordance with the so-called attractor function defined by $a(x_k^{(i)}, y_k)$. In other words, given $p(y_k|x_k^{(i)})$, to move the particles to regions where the importance random weights are higher, $a(x_k^{(i)}, y_k)$ must be chosen in order to guarantee the inequality $\tilde{w}_k^{(i)} \geq w_k^{(i)}$ for all $i$.

For each pair $(x_k^{(i)}, w_k^{(i)})$, the main idea is to compute the corresponding pair $(\tilde{x}_k^{(i)}, \tilde{w}_k^{(i)})$ by using the transformation $r(p(y_k|x_k^{(i)}), a(x_k^{(i)}, y_k); w_k^{(i)})$, the so-called restoration function, that depends on the structure of the importance sampling density $p(y_k|x_k^{(i)})$ and on $a(x_k^{(i)}, y_k)$.

In this way, by using the observed signal $y_k$, the restoration function is applied to obtain the restored observed values for each evolved particle $x_k^{(i)}$, $i = 1,..., P$, i.e.

$$
\tilde{y}_k^{(i)} = r(p(y_k|x_k^{(i)}), a(x_k^{(i)}, y_k); w_k^{(i)}).
$$  \hfill (6.6)
Assuming that $h(\cdot)$ is the deterministic model of the observation signal, the new particle position is obtained by

$$\hat{x}^{(i)}_k = h^{-1}(y^{(i)}_k). \tag{6.7}$$

The restored weight is computed applying the original importance sampling density to the corresponding new particle position, or

$$\hat{w}^{(i)}_k = p(y_k|\hat{x}^{(i)}_k). \tag{6.8}$$

In this case, the selection step occurs without the sorting among importance weights $\hat{w}^{(i)}_k$, $i = 1, \ldots, p$, what facilitates the parallelization of the filter.

Figure 6.1 presents a basic visualization of the restoration method assuming that $p(y_k|x^{(m)}_k)$ is Gaussian and that $a(x^{(m)}_k, y_k)$ is a composition of linear functions around $y_k$ that guarantees $\hat{w}^{(m)}_k \geq w^{(m)}_k$ for all $m$.

![Figure 6.1 - Simple representation of the restoration method.](image)

The obtained $r(p(y_k|x^{(i)}_k), a(x^{(i)}_k, y_k); w^{(i)}_k)$ exemplified in Figure 6.1 makes a very significative correction in the positions of the particles located at the tail of $p(y_k|x^{(i)}_k)$. Additionally, it is easily seen from the picture that, given $p(y_k|x^{(i)}_k)$, the new distribution $p(y_k|\hat{x}^{(i)}_k)$ of particles depends exclusively on the joint properties shared by the functions $p(y_k|x^{(i)}_k)$ and $a(x^{(i)}_k, y_k)$.
The following algorithm shows the restoration method at the time instant $\tau_k$:

---

Restoration Method

- For $i = 1, \ldots, P$
  - Compute $\tilde{y}_k^{(i)}$ via (6.6)
  - Compute $\tilde{x}_k^{(i)}$ via (6.7)
  - Compute $\tilde{w}_k^{(i)}$ via (6.8)

---

6.4.1 Experimental Results

For demonstration purposes, the generic bootstrap algorithm with the restoration method will be applied to data artificially generated by the nonlinear, non-Gaussian model suggested by Gordon, Salmond and Smith (1993) and Kitagawa (1996)

\[
X_k = \frac{1}{2} X_{k-1} + 25 \frac{X_{k-1}}{1 + X_{k-1}^2} + 8 \cos(1.2k) + \Delta W_k \quad (6.9)
\]

\[
Y_k = \frac{X_k^2}{20} + \Delta V_k \quad (6.10)
\]

for an initialization $x_0 \sim \mathcal{N}(0, 1)$, and with $\Delta W_k \sim \mathcal{N}(0, 10)$ and $\Delta V_k \sim \mathcal{N}(0, 1)$ being mutually independent white Gaussian noises.

Based on (6.9), (6.10) and Figure 6.1, a suggested attractor function $a(x_k^{(i)}, y_k) = a$ is the composition of linear functions around $y_k$, that is,

\[
a = \begin{cases} 
N \delta^{-1} [x_k^{(i)} - \beta_1] & \text{if } \beta_1 \leq x_k^{(i)} < y_k \\
-N \delta^{-1} [x_k^{(i)} - \beta_2] & \text{if } y_k \leq x_k^{(i)} < \beta_2 
\end{cases}, \quad (6.11)
\]

where

\[
\beta_1 = y_k - \delta \quad \text{and} \quad \beta_2 = y_k + \delta, \quad (6.12)
\]
$N$ is the normalization factor of $p(y_k|x_k^{(i)})$, and $\delta = 2.0$ is a linear coefficient that satisfies the relation $\tilde{w}_k^{(i)} \geq w_k^{(i)}$ for all $i$. Therefore, the restoration function is defined as

$$
    r(w_k^{(i)}) = \begin{cases} 
        N^{-1} \delta w_k^{(i)} + \beta_1 & \text{if } \beta_1 \leq x_k^{(i)} < y_k \\
        -N^{-1} \delta w_k^{(i)} + \beta_2 & \text{if } y_k \leq x_k^{(i)} < \beta_2 
    \end{cases}
$$

(6.13)

Moreover, based on (6.10),

$$
    x_k^{(i)} = \begin{cases} 
        -\sqrt{20|\tilde{y}_k^{(i)}|} & \text{if } x_k^{(i)} < 0 \\
        \sqrt{20|\tilde{y}_k^{(i)}|} & \text{if } x_k^{(i)} \geq 0
    \end{cases}
$$

(6.14)

A comparison between the effects of the position corrections made by the traditional resampling and restoration methods can be seen in Figure 6.2. At time instant $k$, the initial random measure $\{x_k^{(i)}, \tilde{w}_k^{(i)}\}_{i=1}^{100}$ represented by $\bullet$ is transformed into the new pair $\{x_k^{(i)}, \tilde{w}_k^{(i)}\}_{i=1}^{100}$ represented by $\circ$. In both cases, the effective sample size (LIU; CHEN 1995, LIU 1996) increased. The evolution of the measures is described at time instant $k+1$ and, in this case, the results showed that the restoration method was able to make a better reconstruction of the likelihood function than in the resampling.

FIGURE 6.2 – Comparison of the corrections and evolving effects for the traditional resampling and the restoration method.
To compute the updating of the importance weights, the suggestion is that, at the end of the algorithm iteration \( k - 1 \), there exists an initialization such that \( w_k^{(i)} = P^{-1} \) for all \( i \), as presented in Figure 6.1. Thus, after evolving the random measure \( \{x_k^{(i)}, w_k^{(i)}\}_{i=1}^P \), the weighted state estimate is assigned according to equation (6.1). The restoration method is then applied in order to obtain \( \{\tilde{x}_k^{(i)}, \tilde{w}_k^{(i)}\}_{i=1}^P \), the set with increased effective sample size. This procedure is called Sampling Importance Restoration (SIRe) method. Assuming \( P = 250 \), Figure 6.3 presents a comparison between the real state and the typical results obtained by SIRe method for the cases where the estimates are made before (SIRe - bef) and after (SIRe - aft) the correction of the weights.

![Figure 6.3 - Typical state estimation of the generic bootstrap filter with the restoration method.](image)

The idea now is to compare de filter performance of SIRe methods with the results obtained by the bootstrap filter. Given \( x_0 = 0.10 \) and \( P = 250 \), the comparison was made for the time range \( k = 1, 2, \ldots, 50 \) with 100 different initializations of the noises. The estimation performance of the filters are given by the square root of the mean-square errors (RMSE) at each \( k \), as presented in Figure 6.4.

The results showed that SIRe - bef has practically the same estimation performance as that obtained by the bootstrap filter. However, as presented in Figure 6.3, SIRe - aft smooths the estimates, what improved the filter performance in RMSE viewpoint in this specific case.
FIGURE 6.4 – Comparison of the estimation performance of the bootstrap with SIR methods.

Although the restoration method presented promising results when compared to the traditional resampling scheme, deeper studies about the behavior of the estimates after moving the particles should be made. Future works must be focused on the obtention of possible analytical forms to analyze the effect of the attractor functions in the quality of the estimates.
7 Conclusions

7.1 Main Aspects of the Thesis

This thesis, in a general sense, studied the performance and the applicability of MC methods approximating the solution of the nonlinear filtering problems parameterized according to Feynman-Kac notation \cite{del-moral2004}. The main motivations for the results were their use in the Control Problem which, among other things, requires the deep study of the stochastic modelling, the theoretical form of the nonlinear filtering, beyond the transcription of the resultant multi-dimensional integrals to friendly forms to be applicable to fast electronic systems. The obtained results can just be divided into three stages: 1-) the stochastic modelling of a physical system, 2-) the theoretical solution of the nonlinear filtering, and 3-) the MC-based approaches to solve the nonlinear filtering problem from the numerical viewpoint.

In the former subject, this work defends the hypothesis that every physical system should be modelled according to the continuous-time approach, once the main results in Physics are defined via the use of continuous-time differential equations. External disturbances and even uncertainties in the modelling provide the randomness to the solution of the designed model, implying in the use of SDES. It was shown, recalling a main result from \cite{wong1965} and \cite{clark1973}, that a physical system is represented by Fisk-Stratonovich SDE, though Itô approach is better to manipulate the mathematical operations. These equations in general do not have analytical solutions and the solution properties just can be studied by using numerical methods, such as the Euler-Maruyama...
CHAPTER 7. CONCLUSIONS

and the Milstein schemes. It was also demonstrated, just for clarifying, in what cases the approach of discretizing a deterministic differential equation adding a white noise are valid, once these results are particular cases of the discretization of SDEs.

The second subject gives the main results presented in the classical nonlinear filtering theory, mainly the ones whose representation is developed using unnormalized conditional expectations. This form to present the results facilitates the use of Feynman-Kac notation to solve nonlinear filtering problem. The continuous-time theory was represented by Zakai equation (ZAKAI, 1969) applied to Kallianpur-Striebel formula, and the pathwise or robust formulation (CLARK, 1973)-(DAVIS, 1981b)-(PARDOUX, 1981). Motivated by applications in aerospace systems, one of the contributions of this thesis was to rewrite the robust formulation for a more general case where the observation process contains non-standard Brownian motion, i.e. diagonal covariance matrix with elements different of one. The Bayesian approach was discussed and developed when the discrete-time theory was presented. The difference between both approaches is that in the former the filtering equations are presented explicitly in contrast with the last one.

The latter subject is focused on MC-based methods to solve the continuous-time filtering problems using computers. This subject can be divided in the following two steps: 1-) the numerical solution of Kallianpur-Striebel formula using Zakai or the pathwise equations as the unnormalized representations, and 2-) the numerical solution of the filtering problem using the traditional Bayesian approach.

Aiming at solving numerically the continuous-time nonlinear filtering problem, MC integration was then applied to the integrals given by the unnormalized conditional expectation presented in Kallianpur-Striebel formula. After rewriting this classical approach using the Feynman-Kac notation, it is then possible to implement numerically the conditional expectation integrals using MC methods, as suggested by Talay (1982), Talay (1984) which discretized the Girsanov exponential with Zakai equation and by Davis and Wellings (1980) using the robust representation. Both approaches imply in the development of Monte Carlo filters using independent trajectories simulated by Euler-Maruyama
or Milstein schemes. For the particular case related to the solution of Zakai equation, a branching particle scheme (CRISAN, 2003)-(JACOB; YONEYAMA, 2004) was developed and the obtained results outperformed the extended Kalman filter applied to a stable or unstable Ornstein-Uhlenbeck process with saturation in the observation signal.

For the same system used by the Monte Carlo filters solving Zakai equation, the study of the numerical solutions of the pathwise filter presented a result still not reported by the works of Davis (1981a), O’Loghlen and Wright (1982) and Souza (1992): the numerical estimates diverged when the signal-to-noise ratio was low. Results presented by Jacob and Yoneyama (2005a) showed that the variance of the weights increased dramatically when the signal simulated by a stable Ornstein-Uhlenbeck process is near zero and disturbed by a significative noise amplitude, what seems that the unfavorable noise amplitude might make the filter perform badly. A preliminary solution improved the estimates by running the particle resampling at time instants determined by the effective sample size.

The study of this divergence problem in the pathwise filter was extended in this thesis. The divergence in the estimates occurs because the robust notation of the solution of the nonlinear filtering approach is computed by simulating an alternative SDE with a drift which depends on the observations and which can become unstable. Particularly for the benchmark used, when the signal and the observation processes were close to zero, the resultant drift term made the alternative SDE unstable, what accumulated significant errors to the numerical simulation of the solution.

The Bayesian approach was implemented in accordance with the traditional literature (DOCCET; FREITAS; GORDON, 2001b), except that now the equations were rewritten under the scope of Feynman-Kac notation. The main concern of the this subject was to develop an alternative approach to the particle resampling, a stage in the filtering that is used to decrease the particles degeneracy and, in some sense, to improve the convergence rate of the estimates. This approach, however, brings difficulties to the parallelization of the MC algorithm (BOLIC, 2004), what is not expected in the implementation of filters for real-time applications. The restoration method proposed here moves the particles towards
higher regions in the likelihood function. This open method is suitable for parallelization and, in some sense, decreases the particle degeneracy.

7.2 Future Developments

The study of the solution of the filtering problem using computers can contain a never ending number of unanswered questions depending on how the set of specifications are disposed by the Control Problem, once there are still limitations related to the solution of SDEs and the nonlinear filtering problem under less restricted constraints, the convergence of the discretization schemes for a more general class of unstable systems, the convergence rate of the particle filters for a broader class of systems and approaches for enhancing the filtering performance, among other opened subjects involving some areas of the mathematical sciences.

Specifically for the pathwise filter, the idea to make the use this approach more popular for practical applications is still a challenge, once this method is more sophisticated and requires additional computations to be implemented when compared, for instance, with the branching particle filter. This thesis presented results that suggest the study of the behavior of the estimates when constraints with respect to the simulations of the alternative SDE are imposed. Performance studies comparing the branching particle filters and the pathwise filter seem to be reasonable for clarifying and justifying the use of one or other filter for a given class of problems.

The enhancement of particle filters for real-time applications is other challenge, once the traditional methods for minimizing the effects of the particles degeneracy are difficult to be parallelized. This thesis proposed the basic concepts of the opened approach called restoration method. The idea of using an attractor function to move the particle towards regions of higher likelihood seems reasonable and promising. However, there are not analytical results stating about the effects of these movements on the performance of the estimates.
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11. RESUMO:
This thesis is focused on two basic aspects of the Control Problem: the stochastic modelling of physical systems, and Monte Carlo-based numerical approximation of the nonlinear filtering problem solution. In the first topic this thesis concerns about clarifying some issues in the mathematical modelling of continuous-time systems with Brownian motion. The hypothesis that physical systems should be modelled in continuous-time approach is defended, once the main results in Physics provide solutions for dynamic systems via continuous-time differential equations. It was shown, recalling a main result from the 1960’s that a physical system is represented by Fisk-Stratonovich stochastic differential equation, though Ito approach is better to manipulate the mathematical operations. The required conditions for implementing these equations in computers were also studied by using Euler-Maruyama and Milstein schemes of discretization. In the second topic a unified treatment of the available Monte Carlo methods solving the nonlinear filtering problem for continuous and discrete-time modelling is presented with sufficient emphasis on basic applications enabling the engineer to use results provided by the theory. This topic is branched in the study of the theory of nonlinear filtering problem in continuous and discrete-time approaches, and in the investigation of the aspects of Monte Carlo-based numerical solutions approximating un-normalized conditional expectations, as those given by the classical Kallianpur-Streibel formula and its derived robust representation. Investigations showed that the estimates obtained via numerical approximations of the robust representation, or pathwise filter, might accumulate errors when the observation makes this filter alternative equation unstable, a limitation of the method. Another result of this thesis refers to the implementation of Monte Carlo filters using Bayesian representation for discretized models. Although Monte Carlo methods are attractive due to their facility of parallelization, their main drawback is the degeneracy phenomenon of the particles. The traditional resampling scheme solves the problem, but it difficulties the parallelization of the algorithm. The restoration method was then proposed to move the particles towards higher regions in the likelihood function, given information about the model parameters. This open method, in some sense, might decrease the particles degeneracy.

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